Contributions
to the
History of Indian Mathematics

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Editors

Gérard G. Emch

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Preface

The first Joint India-AMS meeting in Mathematics was held in Bangalore in December 2003. One of its themes was the "History of Indian Mathematics". Two sessions on this theme were organised jointly by Gérard G. Emch from the U.S., and R. Sridharan from India. These sessions were held on the 18th and the 20th of December.

The speakers at these sessions covered a wide spectrum of topics ranging from Vedic Prosody and ancient Buddhist logic at one end to the contributions of Srinivasa Ramanujan and Indian contributions to Quantum Statistics at the other.

The lectures were enthusiastically received and it was felt that a volume based on these lectures in detail might be useful to the mathematical community; we thus formed an editorial committee to compile such a volume. Articles were invited from the speakers and were refereed.

This volume, which is the outcome, begins with an overview[1] of the subject and is divided into three sections.

The first section which deals with the ancient period has two articles, one on Vedic Prosody and the work of Piṅgala and the other on Buddhist Logic.

The next section which discusses the mathematics of the classical and medieval periods begins with two articles, one on the work of Brahmagupta on *Bhāvanā* and its applications, another on the contributions of Bhāskara II to the mathematics of *Karaṇī* or surds. The next article is on the use of power series techniques by the medieval Kerala School of Mathematics. The next two articles focus on the nature of algorithms in Indian Mathematics and Astronomy. The final article of this section is on the notion of proofs in Indian Mathematics and the tradition of *Upapatti*s in Mathematics and Astronomy of India.

[1] The overview is based on a lecture delivered by one of the editors at the Nehru Centre, Mumbai, during September 2002 and is to be published by the Nehru Centre in "Science in India: Past and Present", 2005 (in Press). We are grateful to Mr Sahani, the Executive Secretary of the Nehru Centre for allowing us to include it in this volume.

The third section is devoted to the modern period. The first article points to some surprising contributions of Srinivasa Ramanujan on partial fractions while the second surveys the history of some of the contributions of Indian mathematicians to Quantum Statistics.

The editors are grateful to the organisers of the AMS-India conference for their hospitality and to the contributors for their enthusiastic response. We are very happy to acknowledge the generous help of C. S. Seshadri and his colleagues at the Chennai Mathematical Institute (in particular C. S. Aravinda and V. Balaji) in making this volume possible. We are particularly thankful for the immense task accomplished by of P. Vanchinathan who prepared the camera-ready copy for the publisher.

We thank J. K. Jain of Hindustan Book Agency whose active support and interest has made our endeavour a pleasure.

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Mathematics in Ancient and Medieval India *

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Baruch Spinoza (1632-1677), that gentle philosopher of the seventeenth century laid down as his credo "Sedulo curavi, humanas actiones non redere, non lugere, neque detestari, sed intelligere". He says that when he sets out to interpret the thoughts and history of a bygone age, he obeys the above injunctions set by him for himself – not to ridicule, not to mourn, never to detest, but try to understand – an excellent piece of advice to anyone who interprets ancient writings. I hope I abide by Spinoza's admonitions.

Indian contributions to decimal systems and place value:

Some concepts are so fundamental that they get absorbed in the general thinking of all the peoples of the world and it is indeed hard to pinpoint a specific period of time for their birth. One such is the basic idea of introducing the place value system and the number zero, the credit for which goes to Ancient India. The notion of the place value has become 'obvious' and is now learnt even by young children at an early age. But, to have a true appreciation of the depth of this notion, one has merely to look at the clumsy Roman numerals and note that, not too long ago, these were very much in vogue in the West, in spite of their cumbersome notation. It has been held by a few of the Western scholars that the Arabs were responsible for the decimal notation even though the Arabs themselves clearly indicated that they were merely conveyers of the wisdom of India to the West. But it is now generally accepted that

*Lecture given at the Nehru Centre, Mumbai, on 21st September 2002. To be published by the Nehru Centre in "Science in India: Past and Present", (2005, in Press) the decimal notation is indeed a contribution of India, though, fixing an exact date for this invention is extremely hard. A possible guess could be 100 B.C. Of course, large numbers have always been favourites of Indians at all epochs and are mentioned in Vedic literature, in the epics like the Rāmāyaṇa and the Mahābhārata. A true assessment of the immensity of this great idea of the decimal system and the invention of zero can be found in the following quotation from Professor G.B. Halstead ("On the foundation of the technique of Arithmetic" Chicago, 1912). "The importance of the creation of the zero-mark can never be exaggerated. This gives to airy nothing not merely a local habitation and a name, a picture, a symbol, but helpful power; it is the characteristic of the Hindu race whence it sprang. It is like coining Nirvana into dynamos. No single mathematical creation has been more potent for the general on go of intelligence and power".

The mathematical sophistication of the Harappan culture:

As we know, the Indus Valley civilisation (now, more properly termed the Harappan civilisation) dates back to 2500-3000 B.C. This was a city civilisation and hence, obviously, for town planning, a knowledge of practical geometry must have been a must. Apart from this, we have clear evidence of the geometric sophistication of this ancient civilisation. The patterns on the pottery excavated at these sites show that the artisans were very well acquainted with the concept of circles and well versed in the use of dividers for scratching on the surface of pottery. The motif of triangles is also common on pottery. Apart from the use of such elementary geometry, a sophisticated system of weights and measures used by them shows that the civilisation was well acquainted with basic arithmetic. One of the highlights of this civilisation was its knowledge of brick building technology. The most ideal ratio of the length, breadth and the thickness of a brick to have efficient binding strength was known to the Harappan people.

The vedic period and the Śulva geometry:

There is a huge gap in our knowledge of the interim between the end of the Harappan civilisation and the beginnings of the vedic civilisation. Whatever happened during this period, it is safe to assume that the skills and the culture of the earlier period could not all have disappeared and it is indeed reasonable to postulate that there was a basic continuity in many human endeavours. In particular, the technology of brick building during the Harappan age was perhaps put to good use for the construction of the sacrificial altars during the vedic period. (Let us remember that culture sometimes is defined as a state of the mind produced by things forgotten.)

Mathematics of the vedic period consists of those geometric techniques needed for the construction of the Vedis (altars) and Agnis (fire places) described by the Brāhmaṇas for the obligatory (nitya) and optional (kāmya) rites. This knowledge is contained in Śulva Sūtras which are a part of the so called Kalpa Sūtras (more particularly of the Śrauta Sūtras) attached to the Vedas as Vedāṅgas ("limbs" of the Vedas). There are at least nine extant Śulva Sūtras: Laugākṣi, Mānava, Vārāha, Baudhāyana, Vādhūla, Āpastamba, Hiraṇyakeśin, Kātyāyana and Maitrāyaṇa. The date of the Śulva Sūtras is purely conjectural except that from their language, it is clear that they are pre-Pāṇinian, and that 800 B.C. would be a probable date for the codification of the Sūtras (one must remember that the vedic rituals must have preceded even the Vedas so that the Śulva tradition must be much older than the age of their codification).

The word Śulva means a measuring rope or a rod (though the Sūtras themselves do not use this; rajju is used for a measuring rod). We shall give a very brief sketch of the contents of the Śulva Sūtras. We begin most importantly with the theorem of the square on the diagonal of a right angled triangle, popularly attributed to Pythagoras. A point to remember is that the Śulva Sūtras deal with right angled triangles not for their own sake but as two halves of a square or a rectangle when cut by a diagonal. While it is true that various particular integral values of the sides of a right angled triangle like 3, 4, 5 were known to all the ancient civilisations of the world, the real geometric significance of the theorem was perhaps first realised by the altar building vedic priests. It is believed that the theorem was well known to the Indians at least by 800 B.C. which is the estimated date of the oldest Śulva Sūtra, namely, that of Baudhāyana. Śatapatha Brāhmaṇa gives 36 units as the length of the east-west line segment (prācī) of the so called Mahā vedi, and 30 units as one of the north-south sides and the prācī and half the side, making a right angled triangle with sides 36, 15 and 39. The Śulva geometers were well aware of the notion of similarity of figures; (the Śatapatha Brāhmaṇa mentions the mode of increasing the size of the vedi, fourteen fold, to accommodate the 101st performance of the sacrifice). The Sautrāmaṇī altar, for example, uses a right angled triangle whose sides are $5\sqrt{3}$, $12\sqrt{3}$, $13\sqrt{3}$ and the Aśvamēdha altar involved a triangle with sides of $15\sqrt{2}$, $36\sqrt{2}$ and $39\sqrt{2}$. Thus irrational numbers occurred very naturally, the Vedic priests being interested in constructing altars whose areas were given multiples of the area of another altar. Many rational and integral solutions of the Pythagorean equation $x^2 + y^2 = z^2$ were also quite well known to the Śulva geometers. For example, the triples (3, 4, 5), (5, 12, 13), (7, 24, 25), (8, 15, 17), (12, 35, 37) are explicitly used.

The question whether the Śulvakārās had a *proof* of the Pythagoras theorem can be answered in the following way: Their approach to geometry was purely functional and they did not have the deductive approach to geometry as the Greeks had. The Śulvakārās looked upon geometry as a necessity and a tool and their approach was therefore empirical.

Among the many simple geometrical results that the Śulva Sūtras mention, are the various properties of the diagonals of a rectangle, constructions of isosceles right angled triangles, parallelograms, rhombi, construction of a trapezium similar to another and so on.

As one of the most remarkable statements found in the Śulva Sūtras, I would like to highlight a rational approximation to $\sqrt{2}$ which occurs in the Baudhāyana, Āpastamba and Kātyāyana Śulva Sūtras. The approximation is

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{(3)(4)} - \frac{1}{(3)(4)(34)},$$

which is correct up to five decimal places. It is indeed remarkable, whereas Greek geometry came to a grinding (though temporary) halt when the Greeks met with the phenomenon of irrationality of numbers for the first time, Indian thought went ahead, quite happy to accommodate the irrationals. (Incidentally, a commentator by name Rāma who lived in the middle of the 15th century A.D., in a place called Naimiṣā near modern Lucknow, improved upon this approximation and obtained

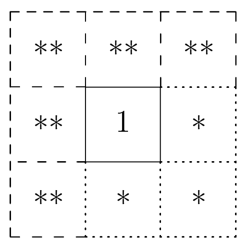
$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{(3)(4)} - \frac{1}{(3)(4)(34)} - \frac{1}{((3)(4)(34)(33)} + \frac{1}{(3)(4)(34)(34)},$$

which gives a better approximation, correct up to seven decimal places).

The Śulvakārās sometimes had to deal with very sophisticated indeterminate equations in two or even four unknowns for the construction of the altars. For instance, in the construction of Garuḍa Cayana, Baudhāyana explains a procedure to have a Vedi constructed with five layers of bricks, each layer consisting of 200 bricks, covering an area of $7\frac{1}{2}$ sq. purshas and he indicates a construction of this vedi, using square bricks of four different sizes. This leads to a fairly intricate indeterminate equation.

Baudhāyana and Āpastamba give different solutions to this problem.

Another curiosity is a derivation of an arithmetical identity from geometry and is found in Baudhāyana Śulva Sūtra (cf. [5]), where he wishes to construct larger and larger squares starting with a small one, by adding successively gnomons to it. The following geometrical figure illustrates this process, and gives a geometric proof of the identity: $1 + 3 + 5 + \cdots + 2n - 1 = n^2$.



We end our very short sketch of the mathematics of the Śulva period with the following passage which is a numerical curiosity from the Śatapatha Brāhmaṇa (cf. [8, p. 351]) about finding all the divisors of 720.[1]

Now this Prajāpati, the year, there are seven hundred and twenty days and nights, his lights (being) those bricks; three hundred and sixty enclosing stones and three hundred and sixty bricks with (special) formulas. This Prajāpati, the year has created all existing things, both what breathes and the breathless, both gods and men: Having created all existing things he felt like one emptied out and was afraid of death. He bethought to himself "How can I get these beings back into my body? how can I be again the body of all these beings?". He divided his body into two. There were three hundred and sixty bricks in one and as many in

[1] See also I. 164 of the Ṛgveda esp., I. 164.11 and I. 164.48.

the other: He did not succeed. He made himself three bodies–in each of them there were three eighties of bricks: He did not succeed. He made himself four bodies of a hundred and eighty bricks each: He did not succeed. He made himself five bodies–in each of them there were a hundred and forty-four bricks: He did not succeed. He made himself six bodies of a hundred and twenty bricks each: He did not succeed. He did not develop himself seven fold (na saptadhā vyabhavat). He made himself eight bodies of ninety bricks each: He did not succeed. He made himself nine bodies of eighty bricks each: He did not succeed. He made himself ten bodies of seventy-two bricks each: He did not succeed. He did not develop eleven fold. He made himself twelve bodies of sixty bricks each: He did not succeed. He did not develop either thirteen fold or fourteen fold. He made himself fifteen bodies of forty-eight bricks each: He did not succeed. He made himself sixteen bodies of forty-five bricks each: he did not succeed. He did not develop seventeen fold. He made himself eighteen bodies of forty bricks each: He did not succeed. He did not develop nineteen fold. He made himself twenty bodies of thirty-six bricks each: he did not succeed. He did not develop either twenty-one or twenty-two or twenty-three fold. He made himself twenty-four bodies of thirty bricks each. Then he stopped at the fifteenth attempt and because he stopped at the fifteenth arrangement, there are fifteen forms of waxing and fifteen of the waning (of the moon).

(Let me add on my own, one more numerical curiosity based on the number seven hundred and twenty. The set of permutations of six symbols which is seven hundred and twenty constitutes a group, the so called symmetric group of degree six. This group contains fifteen transpositions (those permutations which interchange only two symbols) and fifteen other elements of order two which disturb all the six symbols.)

The contribution of the Jainas:

The Śulva period of mathematical development in India represents also simultaneously the mathematical development brought about by the Jains. Fixing dates for Jaina canonical literature is as difficult as (if not more difficult than) fixing the date for Vedic literature. Once, it was thought that Vardhamāna Mahāvīra, a contemporary of Buddha was the founder of Jainism but it has now been conclusively established that he really was the last of the Tīrthankarās; some of the earlier Tīrthaṅkaras like Ṛṣabha, Bharata being well known figures in Hindu

Puranic literature also. Thus, Jainism which arose as a revolt against the vedic sacrificial practices is perhaps as old as the Vedas themselves or even belongs to an earlier period. The mathematics of the Jains is also as old as the mathematics of the Śulvas (if not older). But most unfortunately, our knowledge of the mathematical contributions of the Jains is only through commentaries on earlier works, the original mathematical works of the Jains having not come to light yet. In fact, there is a genuine need to search for original Jain manuscripts.

A name which stands out as a Jain mathematician of fame is Umāsvāti who belonged to the sect of Śvetāmbara Jains. He most probably lived around 150 B.C. and belongs to the Kusumapura school near modern Patna. We note in parenthesis that the later Astronomer - mathematician Āryabhaṭa I (born 476 A.D.) also belonged to the Kusumapura school. Umāsvāti was a great metaphysician. But he seems to have contributed considerably to mathematics too. His famous work is "Tatvārthādhigama Sūtra Bhāṣya". Here are found the well known approximation $\sqrt{10}$ to π and substantial contributions to the geometry of the circles. The approximation $\sqrt{10}$ to π was used by the Jains from 500 B.C. to 1500 A.D. The Jains came to study the circle in connection with their theory of cosmology .

The Jains should have been well aware of the place value system (their heavy calculations giving circumstantial evidence for this). They had names for very large numbers too and eventually had a concept of infinity, however imperfect.

The Jains were aware of the law of indices: $a^m \times a^n = a^{m+n}$. The Jain work Sthānāṅga Sūtra (c. 300 B.C.) classifies the study of mathematics under ten different heads: parikarma, vyavahāra, rajju, rāśi, kalāsavarṇa, yāvat-tāvat, varga, ghana, varga-varga and vikalpa. Parikarma probably refers to the four fundamental operations of arithmetic namely, addition, subtraction, multiplication and division; vyavahāra probably means applied arithmetic, kalāsavarṇa probably refers to the study of fractions, rajju means geometry (as in the Śulva Sūtras) rāśi means measurements in general, varga means squaring, ghana means cubing and varga-varga may mean raising numbers to higher powers. yāvat-tāvat is the "unknown" x in the ancient Indian mathematics. vikalpa is the name given by the Jains to the theory of permutations and combinations. Even though permutations were known from ancient times (they occur in the prosody of Piṅgala, as also in Suśruta's work "Suśruta Saṃhitā" around 6th century B.C. where mention is made of 63 combinations which can be made out of 6 rasas (tastes)), the Jains, however were the ones who treated the theory of permutations and combinations systematically as a topic in mathematics and whose general formulae are completed by the Jain mathematician Mahāvīra (850 A.D.).

The Bakṣālī manuscript:

In 1881 A.D., in the course of excavations, a farmer came across a mathematics manuscript in birch-bark at a village at Bakṣālī near Peshawar. This manuscript was not complete; only about 70 leaves were available, a greater portion having been apparently lost. The language of the manuscript being Gātha (a modified form of prākṛta) in Śāradā script. This manuscript was later printed and published by the then Government of India, edited by G.R. Kaye, with an introduction (Kaye has now become well known for his bias and prejudice against Indians). This manuscript is a collection of rules along with illustrative examples, devoted to arithmetic and algebra, there being not much of geometry, except for a few isolated problems. Perhaps there was a section on geometry which had been lost. The topics on arithmetic deal with fractions, square roots, problems on profit and loss and problems related to the rule of three. The manuscript (whose date is controversial, Kaye having put it around the 12th century A.D., though Datta believes that it belongs to the 3rd or 4th century A.D.) must itself have been a commentary of a much earlier work. We mention some of the mathematics contained in the manuscript. For example, it finds the sum of a finite number of fractions by reducing them to a common denominator; gives the sum of n consecutive terms of an arithmetic or a geometric progression; much more importantly gives a formula for the determination of approximate values of quadratic surds. This formula is usually attributed to Heron, a Greek mathematician of 200 A.D. An English translation of this Sūtra in question runs as follows:

If the number is a non square, subtract the nearest square number, divide the remainder by twice (the nearest square), half square of this is divided by the sum of the approximate root and this when subtracted will give the corrected root.

Expressed algebraically, this reads that if $A = a^2 + r$, then, approximately,

$$\sqrt{A} = a + \frac{r}{2a} - \frac{(\frac{r}{2a})^2}{2(a + \frac{r}{2a})}.$$

This formula is used to give rational approximations to surds like $\sqrt{41}$, $\sqrt{105}$, $\sqrt{889}$ and $\sqrt{339009}$. Using such approximations, the following kind of problem is for instance solved in the Bakṣālī manuscript: Given the sum of the terms of an arithmetic progression whose first term and the common difference are also given, to find the approximate value of the number of terms of this progression.

There are also problems involving indeterminate equations in the Bakṣāli manuscript. Similar problems are discussed by Bhāskara II (about whom we shall discuss later) in Līlāvatī. One is tempted to conclude that such problems were commonly considered in all works even before Bhāskara.

Chandas Sūtra of Piṅgala and Binary Arithmetic: ([11,12,18,19])

Prosody (Chandas) was one of the basic studies undertaken in India right from ancient times. Indeed, prosody was an important Vedāṅga as is borne by the following śloka.

> Chandaḥ pādantu vedasya hastau kalpo'tha paṭhyate |
> Jyotiṣāmayanaṃ chakṣurniruktaṃ śrotramucyate ||

About Piṅgala, not much is known except that he lived most probably three or four centuries before Christ and that he authored an important work on Chandas written in Sūtra style in eight chapters. (it is not clear whether he lived during the time of the grammarian Pāṇini or not). This work was commented upon by several people (Bharata, Halāyudha, Hemacandra, Jayadeva, Jayakīrti, Kedāra, Yādava Prakāśa, ...) of which the commentary of Halāyudha belonging to the 10th century, called Mṛta Sanjīvañī, is very well known.

In the first seven chapters of his work, Piṅgala adopts the conventional method of metrical analysis of the pādas or verses. However, in the last chapter of his work, he introduces a new method which leads to binary arithmetic. Before explaining this method, let me begin with some basic facts on metres. A metre consists of syllables contained in a foot or *pāda* (there being generally four feet in a stanza). A syllable is of two kinds, either it is light (represented by ⏑ and called laghu) consisting of a short vowel followed by at most one consonant or it is heavy (represented by – and called guru) which consists either of a long vowel, a diphthong or a vowel followed by two or more consonants. The usual method of scanning verses is to divide any verse into units of three syllables and assign names to metres on the basis of the combination of the triplets of syllables. As we have remarked, in the first seven chapters of his work, Piṅgala uses this method of classifying metres. However, in the eighth chapter, Piṅgala introduces a new method. Let me make a remark on a standard convention in prosody. The prosodists ignore the first and the last syllable of a foot[2], so that, for example for the Gāyatrī verse

tat saviturvareṇiyam

whose metrical representation is

– ⏑⏑ – ⏑ – ⏑⏑,

the first and the last syllables are ignored so that the Gāyatrī metre is represented by the following sequence of six syllables.

⏑⏑ – ⏑ – ⏑.

Piṅgala associates to a syllabic foot, a table called *Prastāra* which consists of rows of laghus and gurus, listed horizontally. The idea of Piṅgala is to attach a numerical value to each row in the Prastāra by giving the value 0 to a guru and the value 1 to a laghu. The rule for constructing the Prastāra (for the six syllabic verse) is as follows:

First start with all gurus, (six in number in our example). Next, a new row of gurus is started but with the proviso that under the first guru in the line above a laghu is written instead. This procedure is continued until we reach a row consisting entirely of six laghus

[2]cf. e.g. A.A. MacDonnell "Vedic Grammar", Appendix II, p. 437 for general rules of vedic prosody.

$$\begin{array}{cccccc} - & - & - & - & - & - \\ \smile & - & - & - & - & - \\ - & \smile & - & - & - & - \\ \smile & \smile & - & - & - & - \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \smile & \smile & \smile & \smile & \smile & \smile \end{array}$$

Piṅgala attaches a numerical value to each row as follows. Take for example the syllabic pattern $\smile - - - - -$. The number attached to it is just the binary expansion $1 \cdot 1 + 0 \cdot 2 + 0 \cdot 2^2 + \cdots + 0 \cdot 2^5 = 1$, *increased* by 1. i.e $1 + 1 = 2$. The first horizontal line would then correspond to $0 \cdot 1 + 0 \cdot 2 + \cdots + 0 \cdot 2^5$ of 0 increased by 1, so that it has the numerical value 1. In other words, Piṅgala makes correspond to each row, the corresponding dyadic expansion and increases it by 1. For instance the third row has numerical value 3 and so on and the last row of the Prastāra which is simply

$$\smile\smile\smile\smile\smile\smile$$

which corresponds to the binary expansion $1 \cdot 1 + 1 \cdot 2 + 1 \cdot 2^2 + \cdots + 1 \cdot 2^5 = 63$, has as its value $63 + 1 = 64$. The syllabic pattern for the Gāyatrī verse we started with, according to this numbering system has the value 44. The method of constructing a Prastāra has been further explained in the works of Kedāra (8th century A.D.) and Trivikrama (12th century A.D.) and others. To recapitulate, to each horizontal row of a Prastāra consisting of a syllabic pattern one associates the dyadic expansion by assigning the values 0 and 1 to gurus and laghus respectively and by multiplying them by powers of 2 depending on their position and *add* 1 to get the required number. The most remarkable contribution of Piṅgala are the Sūtra (24 and 25 in Chapter VIII of Piṅgala's work) which show how to get back conversely, from any number, a syllabic pattern in a unique manner so that the procedure just described gives a one-to-one onto correspondence between numbers and syllabic patterns. The Sūtras in question are the cryptic

24. l ardhe,
25. saike g.

Translated, they read "a laghu if divisible by 2; if not add 1 and a guru". Let us interpret this remarkably brief Sūtra and write down for example the syllabic pattern corresponding to 44. Note that 44 is divisible by 2 so that we divide 44 by 2 and we start with a $\smile$, the quotient is 22 which is still divisible by 2 so that we divide by 2 and add one more laghu and $\smile\smile$. The quotient is 11 which is not divisible by 2; add 1 divide by 2 and and put a $-$ so that we get $\smile\smile -$. Since the new number 6 is divisible by 2 we add a $\smile$ and get $\smile\smile - \smile$. The quotient is 3 which is not divisible by 2; we add 1 divide by 2 and put a $-$ so that the pattern is $\smile\smile - \smile -$. The number 2 being divisible by 2 we put a $\smile$ and get the pattern $\smile\smile - \smile - \smile$ and end here. This is indeed the metrical pattern for the foot of the Gāyatrī verse we started with.

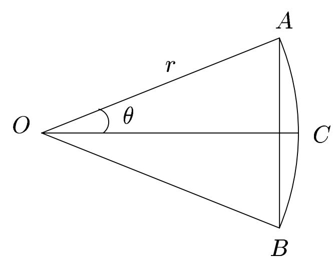
To summarise, by means of a Prastāra, Piṅgala gave an equivalence between the binary representations of numbers and their decimal representations. I would like to remark that the study of Sanskrit prosody initiated by Piṅgala had a continuing tradition in India, the study being taken up by the Jains too.

We note that in Europe, Leibniz, in the late seventeenth century, discovered binary expansions of numbers. When he later came across the Chinese hexagram depictions of Fu Hsi in the "Book of Changes", he interpreted them as representing binary expansions and hence attributed the discovery of binary representation to the Chinese. One has to remember that the Indian contributions to mathematics had not yet reached Europe at that time.

Indian mathematics during the classical age

Āryabhaṭa

Among the mathematicians of the classical age in India, the first important name is that of Āryabhaṭa I (not to be confused with Āryabhaṭa II who lived nearly five centuries later). Not much is known about Āryabhaṭa (from now on when we talk of Āryabhaṭa we mean Āryabhaṭa I) except that in 499 A.D., he wrote his work Āryabhaṭīya, in which he mentions he was then 23 years old, so that one concludes that he was born in 476 A.D. It has been, on the basis of tradition, assumed that he hailed from Kerala, though this is not certain. But, it is certain that he belongs to the Kusumapura school. It is there that he wrote his famous work mentioned above. His work is very concise and hard to understand. Most of it is devoted to Astronomy, though there are 33 ślokas devoted to mathematics. The various subjects dealt with in his work are: the theorem of Pythagoras, methods of extracting square roots and cube roots, progressions, geometrical problems, problems involving solutions of quadratic equations and most importantly, solutions of indeterminate equations of the first degree. Āryabhaṭa's method of solution of indeterminate linear equations was termed by the later mathematicians as Kuṭṭaka (method of *pulverization*). After the discovery of the Bakṣālī manuscript, parts of Āryabhaṭa's work, except those dealing with linear indeterminate equations have lost much of their interest. He was most probably summarising the state of mathematical knowledge of his time. One should, however, make a special mention of Āryabhaṭa's work on his approximation for π and his tabulation of the values of sine. We shall first mention very briefly Āryabhaṭa's discussion of the sine.



We note that in the arc ACB of a circle with centre O shown in the diagram, the chord AB represents $2r\sin\theta$, r denoting the radius of the circle. Āryabhaṭa took $r = 360 \times 60/2\alpha$, where α is a suitable approximation for π, explained in the following śloka:

> caturadhikaṃ śatamaṣṭaguṇaṃ dvāṣaṣṭistathā sahasrāṇām |
> ayutadvayaviṣkambhasyāsanno vṛttapariṇāhaḥ ||

We note the crucial word āsanna in the śloka which means "approximately". Āryabhaṭa defined π *approximately* to be that number which makes the circumference of a circle of diameter 20000 units to be 62832 so that π is approximately 3.1416. It is also very likely, (looking at the above śloka), that Āryabhaṭa was well aware that π is not a rational number. The sine table of Āryabhaṭa consists of the values of $r\sin\theta$ for various angles θ with r as above. It is amusing to note in passing the etymology of the word 'sine' in this connection. In Sanskrit, the arc ACB was called *Cāpa* or *Dhanus*. The chord AB was called *jyā* or *jīvā*. With the Arabs jīvā became jiba and then jaib. Because of the existence of a word in Arabic, meaning heart and sounding similar, the Romans began calling what was originally jyā as sinus which means 'heart' in Latin.

We shall now briefly discuss Āryabhaṭa's work on linear indeterminate equations. i.e. those of the form $ax - by = c$, where a, b, c are integers, x and y being unknowns. The Kuṭṭaka method of solution of such an equation is essentially an algorithm to obtain the greatest common divisor of a and b. This is indeed the most original mathematical contribution of Āryabhaṭa. He was obviously led to consider such equations through problems in Astronomy - those dealing with planetary motions and conjunctions. Āryabhaṭa is very brief (indeed too brief) in his work when he describes this method. There are indeed just two ślokas in Āryabhaṭīya on Kuṭṭaka. These are

> adhikāgrabhāgahāraṃ chindyādūnāgra bhāgahāreṇa
> śeṣaparasparabhaktaṃ matiguṇamagrāntare kṣiptam
> adha upari guṇitam antyayugūnāgra ccheda bhājiteśeṣam
> adhikāgra ccheda guṇaṃ dvi cchedāgram adhikāgrayutam

These ślokas present many difficulties in translating and have led to a lot of different interpretations. B. Datta (1932) has given a translation based on the interpretation of the above ślokas by the commentator Bhāskara I (625 A.D.) of Āryabhaṭa.

The Kuṭṭaka method has been very popular with the later mathematicians of India like Bhāskara I, Mahāvīra, Āryabhaṭa II, Bhāskara II, and right up to the middle ages. There is in fact a text book from South India called Kuṭṭākāra Śiromaṇi by Devarāja. One also has to mention that in the solution of second degree indeterminate equations considered first by Brahmagupta, Kuṭṭaka plays an important role.

The Kuṭṭaka method solves the problem of determining an integer which when divided by two integers a and b leaves remainders r_1 and r_2. A generalization of this problem for several integers $a_1, \ldots, a_n$ and corresponding remainders $r_1, \ldots, r_n$ was also considered by Āryabhaṭa. The Kuṭṭaka method applies to this problem too. We note that this latter is just the classical Chinese Remainder theorem.

Āryabhaṭa I was followed by Varāhamihira, an astronomer of great repute who wrote in 505 A.D. his great work "Pañca Siddhāntikā", expounding the various prevalent astronomical doctrines named Pauliśa, Romaka, Vasiṣṭha, Saura and Brāhma. Among these, he singles out the Saura as the only correct one. Varāhamihira also wrote the famous treatises "Bṛihajjātaka" and "Bṛhat Saṃhitā", dealing with various aspects of Astronomy. Incidentally he seems to be the first astronomer in India who knew the need for the correction due to the precession of the equinoxes.

Brahmagupta

Brahmagupta is perhaps the most remarkable mathematician of the classical age in India. His contributions to mathematics are still remembered and with admiration. He was born in 598 A.D. in Sind (now in Pakistan) and belongs to the Ujjain School, from where he wrote his famous work "Brāhma Sphuṭa Siddhānta", in his 30th year, under the reign of the Śaka King Vyāghramukha. Brahmagupta's idea in writing his work was to bring up to date an older astronomical work entitled "Brahma Siddhānta". This great work of Brahmagupta on Astronomy has more than four chapters devoted to pure mathematics. Among the commentaries on Brahmagupta's work, one should mention especially, that of Pṛthūdaka Swāmi (860 A.D.).

It is interesting to note that Indian Astronomy became known to the Arab world only through Brāhma Sphuṭa Siddhānta. Khalif Abbasid Al Mansoor (712–775 A.D) founded the city of Baghdad (on the bank of Tigris) and made it a centre of learning. A scholar from Ujjain was invited at his initiative in 770 A.D. to explain Indian Astronomy. Through the Khalif's orders Brahmagupta's work was translated into Arabic by Al Fazari and named "Sind Hind" or "Hind Sind". It is thus that the Arab world became cognizant of Indian Astronomy.

We shall pick two brilliant contributions of Brahmagupta, one in Algebra and the other in Geometry which clearly reveal the stature and sophistication of this first rate mathematician.

(a) Varga Prakṛti (Solution of quadratic indeterminate equations)

It must have been well known to all ancient civilizations of the world that the product of the sum of two squares is again a sum of two squares; this comes from the identity

$$(x^2+y^2)(z^2+t^2) = (xz \pm yt)^2 + (xt \mp yz)^2.$$

Brahmagupta stated for the first time a more general version of the above identity called Bhāvana in his work cited above. It asserts that for any arbitrary (integer) N, we have

$$(x^2 - Ny^2)(z^2 - Nt^2) = (xz \pm Nyt)^2 - N(xt \pm yz)^2. \qquad (*)$$

Bhāvana is an example of the notion of "composition" of quadratic forms invented by Gauss several centuries later. (The above identity $(*)$ can be interpreted as saying that binary quadratic forms of the type $x^2 - Ny^2$ admit "composition". The theory of composition of quadratic forms is still an active area of research in mathematics).

Brahmagupta used Bhāvana as a tool to study equations of the form $x^2 - Ny^2 = k$ for integers x, y for any given positive integer N and a given integer k. Such equations are called Vargaprakṛti (square natured). That Brahmagupta considered quadratic indeterminate equations is itself remarkable, in view of the fact that in Astronomy (as we noticed when we were discussing Āryabhaṭa's work) one comes across only linear indeterminate equations and there is a priori no reason to consider Vargaprakṛti, unless out of sheer mathematical curiosity. This clearly shows the working of the mind of a pure mathematician, who has the penchant for picking up significant problems. Brahmagupta showed, using Bhāvana, that if there is one solution (α, β) of the equation $x^2 - Ny^2 = 1$, then one has an infinity of solutions; indeed, applying $(*)$ one sees that $(\alpha^2 + N\beta^2, 2\alpha\beta)$ is also a solution and one could iterate this procedure to get an infinity of solutions. But Brahmagupta did more. He proved using $(*)$ that if one of $x^2 - Ny^2 = -1$, $x^2 - Ny^2 = \pm 2$ or $x^2 - Ny^2 = \pm 4$ has a solution in integers, then $x^2 - Ny^2 = 1$ has a solution also in integers. Brahmagupta, by this method solves the equation $x^2 - 92y^2 = 1$ ($x = 1151$, $y = 120$ is a solution) and the equation $x^2 - 83y^2 = 1$ ($x = 82$, $y = 9$ is a solution). He however was unable to solve the equation $x^2 - Ny^2 = 1$ in general.

The credit for giving an algorithm to solve the above equation in general goes to an unknown Indian mathematician who lived after Brahmagupta but before Bhāskarācārya II (whose contributions we shall discuss subsequently). Bhāskarācārya II, in his work "Bījagaṇita" (circa 1150 A.D.) describes an algorithm to solve this equation and says "people call this method Cakravāla" (the "cyclic method"). Using the Cakravāla method, he solves the rather difficult equations $x^2 - 61y^2 = 1$ and

$x^2 - 67y^2 = 1$ ($x = 1766319049, y = 226153980$ is the least positive integral solution of the first equation and $x = 48842$, $y = 5967$ is the least positive integral solution of the second).

In 1954, K.S. Shukla of Lucknow University discovered (cf. [15]) in Maharaja's Palace library in Thiruvananthapuram a commentary called "Sundarī" by a certain Udaya Divākara on the work of Bhāskara I, called "Laghu Bhāskarīya", where the same Cakravāla method is described with reference to a certain Jayadeva. The commentary itself dates back to 1073 A.D., so that the Cakravāla method must be anterior to Bhāskara II, at least by a hundred years. The Cakravāla method is a beautiful algorithm which reminds one of Fermat's methods. Curiously enough, Fermat (1601-1665) set up as a challenge to some of his contemporaries like Frenicle, Brouncker and Wallis the problem of finding of an integral solution of the equation $x^2 - 61y^2 = 1$ (They indeed found a solution). Thus began, in the 17th century in Europe the study of such equations which is the genesis of modern Arithmetic Geometry. A. Weil (cf. [21]) has this to say in this connection: "What would have been Fermat's astonishment if some missionary just back from India had told him that his problem had been successfully tackled there by the native Indians almost six centuries earlier?". One must add that Euler who looked at this problem of Fermat, wrongly attributed it to a certain British mathematician named Pell (who had nothing to do with this equation) and called it Pell's equation. His name has since stuck to the equation $x^2 - Ny^2 = 1$ through the mistake of Euler. It should really be called "Brahmagupta equation".

As an example of the prejudice of some of the mathematicians of the West, who would insist on depriving India of the honour of discovering mathematical ideas and try to prove that all mathematics originated in Greece, I quote the reasons of (a very fine mathematician) Van der Waerden to dismiss the claim that Brahmagupta was indeed the first one to study the so called Pell's equation and attributing it to the Greeks.

"1) In Brahmagupta's treatise, the problem of solving Pell's equation is not motivated at all.

2) The main subject of Brahmagupta's treatise is Astronomy. His Astronomical system is based upon the epicycle hypothesis which is a Greek invention. As one of his tools, Brahmagupta uses a table of sines. The Greeks have tables of chords which can be easily trans-

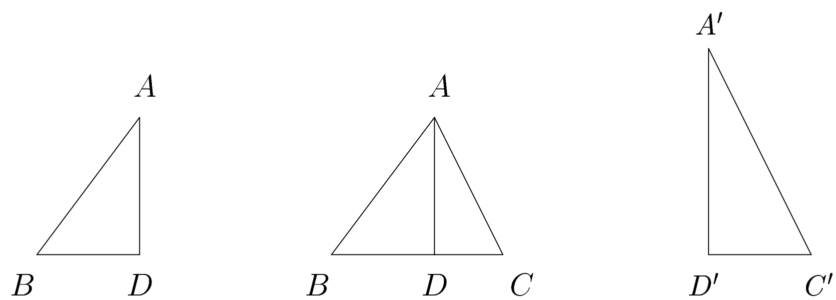
formed into a table of sines. Some eighty years before Brahmagupta, Varāhamihira presents excerpts from five Siddhāntas one of which, the Romaka Siddhānta, was based on the solar and lunar theory of Hipparchus while another Siddhānta is ascribed to "Pauliśā, the Greek". Hence the Astronomy and the mathematical tools of the Hindus at the time of Brahmagupta and before, were largely derived from Greek sources. This makes it even more probable that the Hindu method of solving Pell's equation also goes back to Greek sources.

Let us keep in mind that in history of science, independent inventions are exceptions: the general rule is dependence" (cf. [20]).

(b) Brahmagupta's work on rational cyclic quadrilaterals

I now discuss a typical contribution of Brahmagupta to geometry. As we have already remarked, Śulvakāras were familiar with the construction of right angled triangles with rational sides. It is indeed remarkable that several centuries later, Brahmagupta was still interested in the problem of construction of figures with rational magnitudes, more precisely, in the question of constructing cyclic quadrilaterals with rational sides and diagonals.

We begin with the following remark: Suppose that we have two right angled triangles ADB and $A'D'C'$ with rational sides. We consider another triangle ADC which is similar to $A'D'C'$. By juxtaposing the triangles ADB and ADC along the side AD, we get a triangle ABC with rational sides and rational area. Conversely, any triangle with rational sides and rational area can be obtained by juxtaposing two rational right angled triangles. Indeed, one has to drop the perpendicular from one of the vertices to the opposite side and consider the two right angled triangles thus formed.



This result is attributed to Euler by (cf. [7]) (though no proof is found in Euler's published papers). However this must certainly have been obvious to Brahmagupta, as it is indeed a particular case of Brahmagupta's result on quadrilaterals by looking at a triangle as a degenerate quadrilateral. What Brahmagupta did was much more. He constructed cyclic quadrilaterals with rational sides and diagonals by juxtaposing two right angled triangles in either of the following two ways.

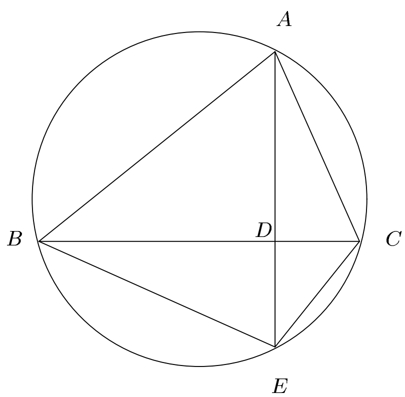


Figure 1

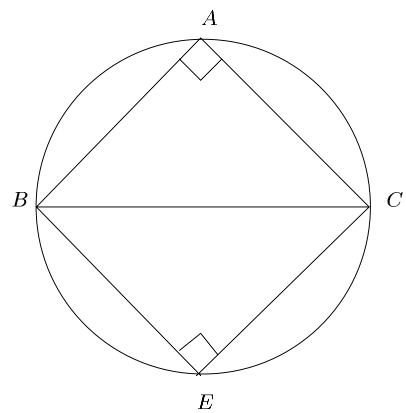


Figure 2

The first figure corresponds to juxtaposing the rational right angled triangle ADB and ADC along AD and producing AD to meet the circumcircle of ABC at E. The quadrilateral $ABCE$ is seen to be one with all sides, the intercepts on the diagonals and hence the diagonals all rational. The other figure corresponds to juxtaposing two rational right angled triangles along their hypotenuses. Various numerical examples were considered by Brahmagupta.

The work of Brahmagupta reached Europe through the translation by Colebrooke in the 19th century. The famous French geometer Chasles was very much impressed by this work and he published an appreciation of this. (cf. [2]). The German mathematician Kummer (cf. [10]) in 1846 considers the general problem for all quadrilaterals, giving a complete solution for the cyclic case. Kummer modifies Brahmagupta's method of construction and shows that one can obtain all cyclic quadrilaterals whose sides, diagonals and area are rational, by employing a juxtaposition of <u>three</u> rational right angled triangles. (It is rather amusing to note that he blames Brahmagupta for not having thought of this solution!) He relates the solution in the possibly non-cyclic case to the existence of rational points of certain elliptic curves.

The questions studied by Brahmagupta on quadrilaterals inspired further study and elaborations by successive generations of Indian mathematicians like Mahāvīra, Bhāskara II and the interest in these problems continued even in the middle ages. For instance, Gaṇeśa (c 1545 A.D.) has results on rational quadrilaterals.

The next few names of mathematicians of note in the classical age in India are those of Śrīdhara (c 750 A.D.) who composed a work on arithmetic called Pāṭīgaṇita and a work on algebra. The latter is no longer extant. We know about the contributions of Srīdhara from the later work of Bhāskara II, who also mentions the name of another algebraist Padmanābha. One of the most celebrated mathematicians after Brahmagupta was Mahāvīra (C 815 A.D.). He was in the court of the Raṣṭrakūṭa king Amoghavarṣa Nṛpatuṅga (who ruled over a part of the present state of Karnāṭaka). Mahāvīra wrote a work entitled "Gaṇitasārasaṃgraha" (C 850 A.D.) which is really the first text book on arithmetic whose material is found even in the present text books in South India. Mahāvīra mainly improved on the work of his predecessors and probably did not make any profound contributions himself.

Bhāskaracārya II

The next celebrated and essentially the last mathematician of the classical period is Bhāskara II who was called Bhāskarācārya in view of the veneration that people had for his learning. Bhāskara II belongs (according to his own statement in his work "Siddhāntaśiromaṇi" written around 1150 A.D.) to Bijjāḍa Bīḍa which has been identified with modern Bijāpur. His work Siddhāntaśiromaṇi is divided into four parts: Līlāvatī, Bījagaṇita (Algebra), Grahagaṇita (Astronomy) and Golādhāya (study of the celestial sphere). Bhāskara prided himself as a poet and wrote all his mathematics in verse form. (To my mind, verse form and mathematics do not go well with each other).

As we already said, Siddhāntaśiromaṇi is essentially a text book, a consolidation of the state of mathematics at the time of Bhāskara II, but his work was systematic and lucid so that Bhāskara was very much respected and termed an "Ācārya".

The part of Siddhāntaśiromaṇi, called Līlāvatī deals with various aspects of arithmetic, whereas Bījagaṇita deals with algebraic, theoretical questions. Līlāvatī has several problems in arithmetic, geometry, quadratic equations, permutations and combinations and Kuṭṭaka. As a typical example, we mention a problem on permutations related to Chandas from Līlavatī:

> Prastāre mitra gāyatryāḥ syuḥ pāde vyaktayaḥ kati |
> Ekādi guravaścāśu kati katyucyatām pṛthak? ||

[The metre Gāyatrī, (as noted before) has six syllables. The number of possible times the long syllable occurs in any pāda in a metre with six syllables is six, the number of times it occurs twice is fifteen and so on, so that the sum total of all its occurrences is 64 (including the case when it does not occur at all. For the Gāyatrī metre with four pādas, the number of all such possibilities is $64^4 = 16777216$)].

There is a tradition that Līlāvatī was the name of Bhāskara's daughter who became a widow at an early age and that Bhāskara wrote his work Līlāvatī to provide consolation for his daughter. It is not clear that this fanciful story has any basis.

In Bhāskara's Bījagaṇita, there are problems which reduce to solving quadratic equations. Occasionally, there is an example of a problem which reduces to an easily solved cubic, or even a biquadratic.

Bhāskara was familiar with the notion of integration as a limit of finite sums. For instance, he calculates the surface area of the sphere, by dividing it into annuli and drawing a system of parallel circles, or by dividing the surface into lunes by drawing meridian circles through a pair antipodal points on the sphere and dissecting each lune into a large number of small quadrilaterals by drawing circles parallel to the equator. Bhāskara also had ideas about differentiation in connection with the so called instantaneous method (tātkālika) by dividing the day into a large number of small intervals and comparing the positions of the planet at the end of successive intervals. The tātkālika gati is essentially the instantaneous velocity of the planet. Bhāskara also knew the addition formula for the sine and special values of the sine function. Bhāskara's infinitesimal approach probably found its fruition in the mathematics of Kerala region in Medieval India which we now proceed to discuss.

Mathematics in Medieval India (cf. [1,12,13,17]

It was believed till nearly a hundred and fifty years ago that creative contributions to mathematics form India came to a halt after Bhāskara II. However, in 1835, Charles M. Whish, a civilian employee of the East India Company sprang a surprise, when he published in the "Transactions of the Royal Asiatic Society of Great Britain and Ireland", a paper entitled "On the Hindu quadrature of the circle and the infinite series for the proportion of the circumference to the diameter exhibited in four Śāstras: Tantrasaṅgraham, Yuktibhāṣā, Karaṇapaddhati and Sadratnamālā". Whish gave an account of the contents of these works and assigned possible dates for these texts, some which have since been confirmed. Since the middle of the 20th century, several Indian scholars have worked on the contributions of the Kerala school during the middle ages and it is now obvious that, at least in Kerala, Indian mathematics did not end with Bhāskara and in fact continued right through the middle ages.

The great pioneer of the Kerala school was Mādhava of Saṅgamagrāma who lived possibly during 1340 A.D. - 1425 A.D. He was usually referred to as a Golavid, an expert on the celestial sphere. Though he did not publish much, (he has works like Veṇvāroha and Sputachandrāpti, both giving rules of thumb for fixing lunar positions and some other works like Agaṇita, Laghuprakaraṇa), Mādhava, however seems to have been a versatile genius and a source of inspiration to his students. We shall talk about Mādhava in connection with a specific problem later. Next in line comes Parameśvara who was a prolific writer on Astronomy and is especially remembered as the author of Dṛggaṇita system of Astronomy. Based on his substantial observations spread over fifty-five years, he made some significant corrections to Āryabhaṭa's astronomical calculations.

The greatest personality of the Kerala school was Nīlakaṇṭha Somayāji, who was a student of Dāmodara (son of Parameśvara and like his father an Astronomer). Nīlakaṇṭha Somayāji lived sometime during 1440 and 1550 A.D., wrote a first rate commentary on Āryabhaṭīya, appropriately called Bhāṣya (reminiscent of the great Mahābhāṣya of Patañjali on the Aṣṭādhyāyī of Pāṇini!). This work must have been composed late in his life since he quotes there from his other works including Tantrasaṅgraham. It may be interesting to note that among his works there is one which is called Sundararāja Praśnottara which records Nīlakaṇṭha's answers to specific questions of a certain Sundararāja, a contemporary scholar from the Tamil country. The respect Sundararāja had for Nīlakaṇṭha bordered on veneration.

Jyeṣṭhadeva was another student of Dāmodara and he authored Yuktibhāṣā, a scientific text written in Malayalam modelled on Nīlakaṇṭha's Tantrasaṅgraham. Yuktibhāṣā is indeed unique in the sense that it is the first text in mathematics in India to state precise mathematical propositions *along* with their proofs. This was composed around 1520 A.D. Karaṇapaddhati was written by Putumana Somayāji around 1730 A.D. This is a work of ten chapters and verses and summarises all the mathematical contributions of the Kerala school in the sixth chapter, often quoting from Mādhava, Nīlakaṇṭha and others.

Sadratnamālā belongs to the period after 1800, its author being Śaṅkara Varma, a ruler of the local principality Kaḍāthanāḍ in north Kerala. This work is merely an exposition of the achievements of the Kerala mathematicians.

Though most of the above texts deal with Astronomy or with some special topics, they always contain sections on mathematics; for example Yuktibhāṣā contains usual mathematical material on arithmetic, fractions, Kuṭṭaka, a preliminary section with a proof of the Pythagorean theorem, approximation of the circumference of the circle by the perimeters of inscribed regular polygons and finally and most importantly an infinite series for $\pi/4$, which is customarily attributed to Gregory and Leibniz. It also contains several results on convergence of series, series for the sine and the cosine and a correct computation of the volume of the sphere, by the method of integration. It has also problems on the geometry of quadrilaterals.

During our discussion of the work of Āryabhaṭa, we mentioned his approximation for π as 3.1416 and his "qualification" that the latter is indeed an approximation. Nīlakaṇṭha raises in his Bhāṣya of Āryabhaṭīya the question "why this qualification?", "why not the precise value?" and he answers "because it is impossible to obtain a precise one", asserting essentially that π is irrational. He also refers to the great Mādhava and says that Mādhava had even a better approximation to π namely 3.1415926536 (this approximation is quoted in Karaṇapaddhati IV, 7). In other words, the mathematicians of Kerala already had a good notion of irrational numbers. Mādhava also seems to have had the idea to look for an "infinite series" for π which would eventually lead to rational approximations. Proceeding in this manner, the Kerala school did produce power series expansions for $\tan^{-1} x$, $\sin x$, $\cos x$ etc. Of course, they did not formally discuss the convergence of these series, but had a clear idea about convergence and in fact constructed more and more rapidly convergent series, and broke infinite series at a finite stage to estimate the error term. Let us remember, all these were achieved at least two centuries before Europe ever came to consider such questions.

We conclude our account of the contributions of the Kerala school, by detailing just one example, which shows the level of sophistication and depth reached by them in analysis. Yuktibhāṣā, discussed earlier contains the following verse which is attributed to Mādhava

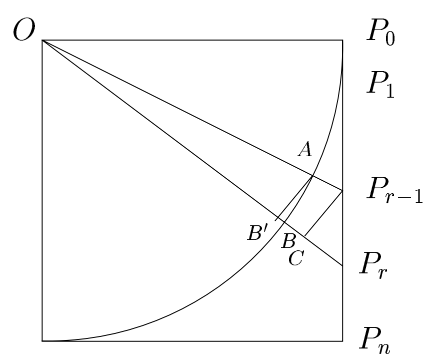
Vyāse vāridhinihate rūpahṛte vyāsasāgarābhihate |
Triśarādiviṣamasaṃkhyā bhaktamṛṇaṃ svam pṛthak kramāt kuryāt ||

which is merely a re-rendering of a verse in Karaṇapaddhati. The content of the verse above is to relate the circumference c of a circle of diameter d in terms of d by the series

$$c = 4d - \frac{4d}{3} + \frac{4d}{5} - \cdots ,$$

or equivalently $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots$, which is the so called Gregory-Leibniz series. We shall present the proof of the above equality due to the Kerala school.

Consider a quarter of a circle of unit radius with centre O inscribed in a unit square as in the following figure:



Partition P_0P_n into n equal parts. Let OP_{r-1}, OP_r meet the circle at A and B. We draw AB' and $P_{r-1}C$ as perpendiculars to OP_r from A and P_{r-1} respectively.

From the similarity of the triangles OAB' and $OP_{r-1}C$, we get, in particular

$$\frac{AB'}{OA} = \frac{P_{r-1}C}{OP_{r-1}}.$$

From the similarity of the triangles $CP_{r-1}P_r$ and OP_0P_r, we obtain

$$\frac{P_{r-1}C}{P_{r-1}P_r} = \frac{OP_0}{OP_r},$$

so that we get

$$AB' = \frac{OA \cdot OP_0 \cdot P_{r-1}P_r}{OP_{r-1}OP_r} = \frac{P_{r-1}P_r}{OP_{r-1} \cdot OP_r}.$$

We note that if n is sufficiently large, the segment AB' tends to arc AB and OP_{r-1} and OP_r are approximately equal, so that we have the approximation

$$\text{arc } AB \sim \frac{1/n}{OP_{r-1}^2} = \frac{1/n}{1+(\frac{r-1}{n})^2}.$$

We therefore have, by using the geometric series $\frac{1}{1+x} = 1 - x + x^2 - \cdots$,

$$\begin{aligned} \frac{\text{circumference}}{8} &= \pi/4 = \lim_{n\to\infty} \textstyle\sum_{r=1}^{n} \frac{1/n}{1+(\frac{r-1}{n})^2} \\ &= \lim_{n\to\infty} \left(\frac{1}{n}\sum_{r=1}^{n-1} 1 - \frac{1}{n^3}\sum_{r=1}^{n-1} r^2 + \frac{1}{n^5}\sum_{r=1}^{n-1} r^4 - \cdots \right). \end{aligned}$$

Remarkably, the Kerala mathematicians knew that for any integer $p \geq 0$, one has that $\frac{1}{n^{p+1}} \sum_{r=1}^{n-1} r^p$ tends to $\frac{1}{p+1}$ as $n \to \infty$. Substituting this in the equation above and pushing the limit inside (which is correct, but needs justification), we find the infinite series expansion for $\pi/4$ namely

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots .$$

In fact, the same method also exhibits the series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots .$$

That $\frac{1}{n^{p+1}} \sum_{r=1}^{n-1} r^p$ tends to $\frac{1}{p+1}$ as $n \to \infty$ was proved in Europe by Roberval in 1634, nearly a century later than the discoveries of the Kerala mathematicians.

As we mentioned earlier, the story does not end here. The Kerala school had several concrete infinite series, they also could manipulate to obtain rapidly convergent series and they even had vague notions of integration and differentiation.

The achievements of the Kerala school may in part be explained by the leadership of that brilliant mind Mādhava. The reason why the discoveries of the Kerala school did not reach the West is perhaps because the Portuguese dominated the West Coast of India and were not interested in the contributions of the natives. It indeed needed the British functionary Whish to bring to the knowledge of the rest of the world these achievements and it further took nearly a century for the Indian scholars to take cognition of this great and yet neglected chapter in the history of mathematics.

Conclusion

To sum up, the contributions of India to mathematics during the ancient, classical and medieval periods have been very noteworthy and many times profound. However, it has taken quite a while for the world to understand and appreciate their depth. There could be several reasons for this. It can in part be attributed to the overwhelming awe the rest of the world had for India's contributions to religion and transcendental philosophy, vis-a-vis the other contributions of India! Yet another reason could be the method, peculiar to ancient India, to hand down knowledge to posterity. Due, perhaps, to the paucity of writing material, important ideas were preserved in the cryptic "Sūtras" which could not be deciphered unless through extensive commentaries by certain distinguished teachers belonging to later traditional schools of learning based on the gurukula system. This sometimes had the effect of hiding the real meaning of the Sūtras from the uninitiated and even if the Sūtras did make sense, their full meaning was not always apparent. This had also the unfortunate effect of obscuring the original date of discovery of a particular Indian contribution, there being a possible ambiguity between the date of the actual discovery and the date of a later commentary on it.

I would like to end this article with a couple of general remarks. The Indian habit of using the verse form to state mathematical problems and propositions has not been of much help in the propagation of mathematics; very often versification could be more of an impediment than help. One might also add that unlike in Greece, where mathematics was held in high esteem for its own sake, in India, scientific thought was generally held subservient to tradition and was not cultivated for its own sake. This, in my opinion, is indeed a weakness in the approach in ancient and medieval India to science, in general, and mathematics, in particular.

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Sanskrit Prosody, Piṅgala Sūtras and Binary Arithmetic

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Abstract

In India, the science of prosody, had its beginnings in the Vedic period and has been held in high esteem, being regarded as a *vedāṅga* or a limb of the *Veda*. The earliest work on prosody was by Piṅgala which is generic and on which all the subsequent works are based. It is an amazing fact that this early work already deals with matters relating to problems of combinatorics. The main aim of this paper is to give a brief description of this work of Piṅgala and discuss in detail the mathematics arising out of it.

§1. Introduction

Classical Sanskrit composition is of two kinds: *gadya* (prose) and *padya* (poetry). Sanskrit prosody, the study of the metrical scanning of Sanskrit poetry, called *chandas śāstra*, has its beginnings already in the vedic times. In his classic work, *Vedic Metre in its Historical development*, first published in 1905, E. V. Arnold ([3]) begins with high praise for the *Ṛgveda* by remarking that *The Rigveda is not a book, but a library and a literature.* At the end of the first chapter, he adds: *To whatever conclusions we may be further led in detail, it must be plain that as works of mechanical art, the metres of Rigveda stand high above those of modern Europe in variety of motive and in flexibility of form. They seem indeed to bear the same relation to them as the rich harmonies of classical music to the simple melodies of the peasant. And in proportion as modern students come to appreciate the skill displayed by the Vedic poets, they will be glad to abandon the easy but untenable theory that the variety of form employed by them is due to chance, or* *the purely personal bias of individuals; and to recognize instead that we find all the signs of a genuine historical development, that is of united efforts in which a whole society of men have taken part, creating an inheritance which has passed through generations from father to son, and holding up an ideal which has led in turn to seek rather to enrich his successors than to grasp at his own immediate enjoyment. If this was so, then the vedic bards are also to be counted amongst 'great men and.....such as sought out musical tunes and set forth verses in writing'.*

To quote another great British Indologist on the Indian contributions to metres, here is what H. D. Colebrooke has to say in his essay entitled *On Sanscrit and Prakrit poetry*, published in "Asiatic Researches" in 1808:xxx (and reprinted in *Essays on History, Literature and Religion of Ancient India* ([9])): *The prosody of Sanskrit is found to be richer than that of any other known language, in the variations of the metre, regulated, either by quantity, or by the number of syllables both with or without rhyme and subject to laws imposing in some instances rigid restrictions, in others, allowing ample latitude.*

A few words about the arrangement of the article: Since a general reader, (for instance a mathematician) to whom this article is addressed, may not be familiar with principles of prosody in general and Sanskrit prosody in particular, this article begins with a rather discursive account of some aspects of vedic prosody and also introduces some basic facts on the rules of Sanskrit prosody in the first few preliminary sections. The main theme of the paper which is the discussion of the classical work of Piṅgala on prosody is taken up, beginning §5. The crucial section is §8 , from the point of view of a mathematician, which deals with the combinatorial aspects of Piṅgala's work.

§2. The beginnings of prosody

As we said earlier, the beginnings of Sanskrit prosody go back to the vedic times. The *Brāhmaṇa*s speak eloquently of the origins of metres, colouring them with mysticism. The study of prosody has been held, right from the early times, with the greatest of esteem; At the end of section 7 of *prapāṭhaka 1* of the *Nidāna Sūtra* ([16]), we have for instance the following stanza:

chandasāṃ vicayaṃ jānan yaḥ śarīrādvimucyate *chandasāmeti sālokyamānantyāyāśnute śriyam*

छन्दसां विचयं जानन् यः शरीराद्विमुच्यते।
छन्दसामेति सालोक्यमानन्त्यायाश्नुते श्रियम्॥

Translated roughly into English the above stanza reads:

One who a has deep knowledge of Chandas, shares, after liberation from his body, the same abode of the Chandas, acquiring eternity, glory and beatitude

Prosody has been described as the feet of the *Veda*s; it is thus one of the limbs of the *Veda* - a *vedāṅga*. In *Pāṇinīya śikṣā*, we have the following verse:

Chandaḥ pādantu vedasya hastau kalpo'tha paṭhyate
Jyotiṣāmayanaṃ cakṣurniruktaṃ śrotramucyate

छन्दः पादन्तु वेदस्य हस्तौ कल्पोऽथ पठ्यते ।
ज्योतिषामयनं चक्षुर्निरुक्तं श्रोत्रमुच्यते ॥

Chandas are the feet of the Vedas, Kalpa the hands, Astronomy the eyes and Nirukta the ears.

The importance of the knowledge of *chandas* for understanding the *Veda*s is emphasised in *Bṛhaddevata* (a text dealing with the gods of the *Ṛgveda*, supposedly written by that ancient venerable vedic seer Śaunaka, verse 136, VIII, ([5]) as follows:

aviditvā ṛṣiṃ chando daivataṃ yogameva ca
yo'dhyāpayejjapedvāpi pāpīyāñjāyate tu saḥ

अविदित्वा ऋषिं छन्दो दैवतं योगमेव च ।
योऽध्यापयेज्जपेद्वापि पापीयाञ्जायते तु सः ।

One who teaches or recites the Veda without having proper knowledge of the applications, the seers, metres and Gods, becomes indeed a sinner.

As one of the earliest references to prosody, we have a verse (I.1.5) in the *Muṇḍakopaniṣad* ([10]), which lists *chandas* as essential for attaining the "lower knowledge", the higher one being that of the *Brahman*.

tatrāparā ṛgvedo yajurvedo sāmavedo atharvavedaḥ
śikṣā kalpo vyākaraṇaṃ niruktaṃ chando jyotiṣamiti

तत्रापरा ऋग्वेदो यजुर्वेदो सामवेदो अथर्ववेदः ।
शिक्षा कल्पो व्याकरणं निरुक्तं छन्दो ज्योतिषमिति ॥

The six *vedāṅga*s which were considered essential for the understanding of the *Veda*s are as mentioned above: *śikṣā*, phonetics; *kalpa*, the knowledge of the sacrificial rites; *vyākaraṇa*, grammar; *nirukta*, etymology; *chandas*, prosody; and *jyotiṣa*, astronomy.

These *vedāṅga*s, whose beginnings can be traced already to the *Brāhmaṇa*s and the *Āraṇyaka*s, did not originally refer to independent branches of knowledge but were only indicated as fields of study, essential for the understanding of the *Veda*s. As time went by, it was realised that there was a real need to develop them as auxiliary subjects associated with the study of the *Veda*s. Hence, independent texts were written in the (mnemonic) *sūtra* style to expound these subjects. ([28]) . It is perhaps worthwhile to mention, by the way, that the *sūtra* style of writing is something unique to Indian literature. A very succinct definition of a *sūtra* is found ([14]) in the *Viṣṇudharmottara purāṇa* and runs thus:

alpākṣaram asandigdhaṃ sāravat viśvatomukham
astobham anavadyaṃ ca sūtraṃ sūtravido viduḥ

अल्पाक्षरं असन्दिग्धं सारवत् विश्वतोमुखं ।
अस्तोभम् अनवद्यं च सूत्रं सूत्रविदो विदुः ॥

A sūtra should have the least number of syllables, should contain no doubtful words, no redundancy of words, should have unrestricted validity, should contain no meaningless words and should be faultless!

The study of prosody which began in the vedic period evolved to apply to classical Sanskrit and to the Prakrit poetry as well and had its continued impact on the poetry of the later ages too.

To quote an example, the *śloka*s of the epics like the *Rāmāyaṇa* and the *Mahābhārata* are derived by and large from the vedic metre *anuṣṭup*. Indeed the vedic metre *anuṣṭup* came to be monopolised by the poets of the classical age. On the other hand, the vedic metres *triṣṭup* and *jagatī* led to metres used by poets and bards at the courts of various kings.

In this connection, it is amusing to note, parenthetically, that the great poet Kālidāsa (who himself is probably the author of a work entitled *Śrutabodha* [21] on classical metres - though this work is attributed by some to Vararuci) employs a vedic metre in a very appropriate context in his great play *Abhijñāna Śākuntalam*.

When Śakuntalā is about to leave the hermitage of the sage Kaṇva to go to meet her husband Duṣyanta, Kaṇva offers a benediction, which is set in the following beautiful stanza in the vedic metre *triṣṭup*, with 11 syllables in each of its four *pāda*s.

amī vedīm paritaḥ klṛptadhiṣṇyāḥ
samidvantaḥ prānta saṃstīrṇa darbhāḥ
apaghnanto duritaṃ havyagandhaiḥ
vaitānāstvāṃ vahnayaḥ pāvayantu

Translated in to English, the stanza reads:

May these sacrificial fires, fixed in their places around the altar, nourished by holy wood, with the darbhā grass strewn around their boundaries, removing sin by the fragrance of the oblations, purify thee!

अमीवेदिं परितः क्लृप्तधिष्ण्याः
समिद्वन्तः प्रान्त सम्स्तीर्ण दर्भाः ।
अपघ्नन्तो दुरितं हव्यगन्धैः
वैतानास्त्वां वह्नयः पावयन्तु ॥

§3. Units of prosody, the syllables

The etymology of the Sanskrit word for prosody *chandas* traces it to various roots, for instance, it can de derived from the root *chad*, which means "to cover"; incidentally, this is not the only possible etymological derivation; there are several other possibilities too! Whatever be the etymology of *chandas* and the consequent derived meanings, it denotes the science of syllables in verses.

A syllable (*akṣara* in Sanskrit) is a vowel with or without one or more consonants. A syllable is called a *laghu* (short), (denoted by l), if it consists of a short vowel followed by at most one consonant. A syllable which is not a *laghu* is called a *guru* (long), (denoted by g). But there is a proviso by which even a short syllable will be treated as long while scanning, when it is followed by a conjunct consonant, an *anusvāra* (a nasal) or a *visarga* (an aspirant). Unlike in classical Sanskrit prosody, where the nature of the syllables is also an important aspect of prosody, vedic metre is governed solely by the number of syllables in a verse, called the *length* of the metre. (a verse is called a *pāda* in Sanskrit), which forms the basic unit of Sanskrit poetry. Verses combine to form a *ṛk*, or a stanza, which is a unit of a vedic hymn.

A stanza consists, generally, of not less than three and not more than fifteen verses. A stanza may consist of metrically identical (*sama*) or metrically different (*viṣama*) verses. Two or three stanzas combine to form a strophe.

The following is an example of a *ṛk* in *gāyatrī* metre (a stanza with three verses each of which has 8 syllables):

agnim īle | puróhitam |
yajñásya de | vam ṛtvijam |
hótāram ra | tnadhā́tamam |

which has the following arrangement of 8 syllables in each of its verses:

$\not g \quad l \quad g \quad g \quad l \quad g \quad l \quad \not l$

$\not g \quad g \quad l \quad g \quad l \quad g \quad l \quad \not l$

$\not g \quad g \quad g \quad g \quad l \quad g \quad l \quad \not l$

(According to one of the rules of prosody, the first and the last syllables of a verse are ignored for scanning purposes.)

§4. Some works on prosody other than Piṅgala's

As we shall notice presently, Piṅgala wrote a definitive work (in *sūtra* style) on prosody, probably around the middle of the third century B.C. As is the case with such definitive works, (for example the *Aṣṭādhyāyī* of Pāṇini), Piṅgala's work systematises and improves upon the work of many earlier authors on the subject. The names of Yāska and the otherwise unknown prosodists like Saitava, Rāta, Māṇḍavya, Tāṇḍī, Krauṣṭiki and Kāśyapa are mentioned as some of those who preceded him. Like Pāṇini once again, who dealt with classical Sanskrit grammar rather than vedic grammar, Piṅgala, though he begins his work with vedic metres, deals for a substantial part with classical metres. It should be remarked that works like the *Chando viciti* (called *Tatva subodhinī*), which is a part of the *Nidāna Sūtra* (which is a *śrauta sūtra* of the *Sāmaveda*, and is supposed, according to some, to have been written by the great Patañjali, who wrote the *Mahābhāṣya* - the "great commentary" - on Pāṇini's *Aṣṭādhyāyī*), *Ṛkprātiśākhya* (written by the venerable Śaunaka), *Sāṅkhyāyana Brāhmaṇa*, associated to the *Ṛgveda* and *Ṛksarvānukramaṇī*, also deal with various aspects of vedic metres. The *Agni purāṇa, Nārada purāṇa, Garuḍa purāṇa*, and the *Viṣṇudharmottara purāṇa*, *Nāṭya Śāstra* by Bharata, and Varāhamihira's *Bṛhat saṃhitā* are some of the fairly old texts which have separate sections dealing with Sanskrit prosody. Subsequent to the classic work of Piṅgala, apart from commentaries on it, like that of Halāyudha (called *Mṛta sañjīvanī* ([11]), that of Yādava Prakāśa ([19]), there have been many authors like Kedāra Bhaṭṭa ([25]), Svayambhū (847 A.D.), Kṣemendra (1100 A.D.) and Gaṅgādāsa (1500 A.D.)([6]) and others ([26]), who have written texts on prosody. As we mentioned earlier, even Kālidāsa is said to be the author of the text *Śrutabodha*, dealing with classical Sanskrit prosody. All of these are heavily influenced by the monumental work of Piṅgala. There have also been many Jain authors who have written on prosody, like the author of Jānāśrayī (6th to the 7th century A.D)([12]), Jayakīrti (1000 A.D.) Jayadeva (1000 A.D.) and that polymath from Gujarat, Hemacandra (1088–1172 A.D.)([7]).

§5. Piṅgala, the author of Chandas Sūtra

As is the case with many of the ancient personages in India, very little is known about Piṅgala himself except that he was highly venerated and referred to as *Piṅgalācārya* or *Piṅgala Nāga*. (*Nāga* in Sanskrit means a serpent and serpents are supposed to be endowed with great wisdom). Some think that he was identical with Patañjali the author of the *Mahābhāṣya*. Ṣadguruśiṣya in his commentary (1187 A.D.) on *Ṛganukramaṇī* refers to Piṅgala as *pāṇinīyānuja* which can be interpreted to mean that Piṅgala was a younger contemporary of Pāṇini or even that he was the younger brother of Pāṇini. Though, conjecturally, it is thought that Piṅgala lived in the middle of the third century B.C.,the precise period of Piṅgala is hard to determine. Most probably, Piṅgala was a younger contemporary of Pāṇini and belonged to the third century B.C. With reference to his place of birth, we are equally ignorant, though it is surmised that he might have been born somewhere on the west coast of India. That he lived near a coast is perhaps obliquely corroborated by the statement in the *Pañcatantra* (2,36) (cf. [18], p. 255) about the manner in which Piṅgala met his death. Stressing the theme that even the meritorious ones can not take it for granted that they are safe from assault, it is mentioned there

chandojñānanidhiṃ jaghāna makaro velātaṭe piṅgalam

Translated into English, it reads *Piṅgala, the repository of the knowledge of metres was killed by a crocodile on the sea shore.* The full verse in fact says that Pāṇini was killed by a lion, Jaimini by an elephant and Piṅgala by a crocodile. Albrecht Weber in his book, "Uber die Metrik der Inder" ([27]), guesses that this enumeration is perhaps in the order of time and therefore Piṅgala probably was later in time than Pāṇini and Jaimini.

§6. Piṅgala's Chandas Sūtra

However uncertain one is about Piṅgala as a man and his life history, his work on *chandas* (in eight chapters, containing 315 *sūtras*) is very much extant and has been commented upon, as we said earlier, by several distinguished authors including Halāyudha (11th century), Yādava Prakāśa (11th century), the latter being the well known teacher of Ramānuja. As we also mentioned, there are several later texts on Sanskrit prosody based on Piṅgala's work, one of the most important one being by Kedāra

Bhaṭṭa (12-13th Century). We note also that in the *Agni purāṇa* ([1]), chapters 327-334 give a summary of the *Chandas Śāstra* as expounded by Piṅgala, beginning with a description of prosody thus:

chando vakṣye mūlajaistaiḥ piṅgaloktaṃ yathākramam

छन्दो वक्ष्ये मूलजैस्तैः पिङ्गलोक्तं यथाक्रमं ।

In Varāhamihira's *Bṛhatsaṃhitā* ([23]) in section 104, which deals with *grahagocarādhyāya* (movements of planets), verse 58, emphasising the rule of prosody (already found in the first chapter of Piṅgala's *Chandas Sūtra*), reads:

prakṛtyāpi laghuryaśca vṛttabāhye vyavasthitaḥ
sa yāti gurutāṃ loke yadā syuḥ susthitā grahāḥ

प्रकृत्यापि लघुर्यश्च वृत्तबाह्ये व्यवस्थितः ।
स याति गुरुतां लोके यदा स्युः सुस्थिता ग्रहाः ॥

Very much like the final syllable in a verse which is deemed long by the rules of prosody even if it is short, a person though of mean birth, and reprehensible in character, becomes respectable in this world, if the planets are favourable.

§7. A brief discussion of Piṅgala's Chandas Sūtra

As we said earlier, Piṅgala's *Chandas Sūtra* contains 315 *sūtra*s distributed over eight chapters. Among these, the *sūtra*s of the first three chapters and the first seven *sūtra*s of the fourth are devoted to vedic metres. As mentioned before, the two basic building blocks of Sanskrit prosody are the *guru* (g) and the *laghu* (l). These correspond to the Greek syllables: *thesis* and *arsis*. From these, the following groups of disyllables can be built:

g	g	–	which in Greek is the disyllable	*spondee*
l	g	–	″	*iambic*
g	l	–	″	*trochaeus*
l	l	–	″	*pyrrhic*

Obviously, the number of trisyllables is eight and are as written below:

Trisyllable	**Greek name**	**Sanskrit name**
g g g	*molossus*	*magaṇa*
l g g	*bacchius*	*yagaṇa*
g l g	*amphimacer*	*ragaṇa*
l l g	*anapaestus*	*sagaṇa*
g g l	*antibacchius*	*tagaṇa*
l g l	*amphibrachys*	*jagaṇa*
g l l	*dactylus*	*bhagaṇa*
l l l	*tribrachys*	*nagaṇa*

In the first chapter of his work, Piṅgala gives the mnemonics *ma, ya, ra, sa, ta, ja, bha, na* to the set of trisyllables written above. Any trisyllable is called a *gaṇa*, so that the trisyllables are denoted, respectively by *magaṇa, yagaṇa, ragaṇa, sagaṇa, tagaṇa, jagaṇa, bhagaṇa, nagaṇa.* These are referred to in the Piṅgala's *Chandas Sūtra* as *aṣṭau vasava iti.* Piṅgala remarks that these gaṇas along with the *guru* and *laghu* form the basis of all prosody.

Many works on prosody, like *Vṛtta Ratnākara* of Kedāra Bhaṭṭa (1150 A.D.), the commentary of Yādava Prakāśa (circa 1050 A.D.) and many other commentators of Piṅgala's work have the following couplet which expresses poetically the pre-eminence of the above ten units of prosody:

myarastajabhnagairlāntaiḥ ebhirdaśabhirakṣaraiḥ
samastaṃ vāṅmayaṃ vyāptaṃ trailokyamiva viṣṇunā

म्यरस्तजभ्नगैर्लान्तैः एभिर्दशभिरक्षरैः ।
समस्तं वाङ्मयं व्याप्तं त्रैलोक्यमिव विष्णुना ॥

The world of speech is enveloped by the ten units ma, ya, ra, sa, ta, ja, bha, na, g and l, like Lord Viṣṇu permeating the three worlds.

(The same statement is also made by Ṣadguruśiṣya in his commentary of *Ṛksarvānukramaṇī.*) We quote another *śloka* given by Kedāra Bhaṭṭa in his *Vṛtta Ratnākara*, which gives a mnemonic for the eight *gaṇa*s:

ādimadhyāvasāneṣu bhajasā yānti gauravam *yaratā lāghavaṃ yānti manau tu guru lāghavam*

आदिमध्यावसानेषु भजसा यान्ति गौरवं ।
यरता लाघवं यान्ति मनौ तु गुरुलाघवं ॥

A rough translation in to English of the above *śloka* reads:

The guru moves into the first, middle and the last position in bha, ja and sa. The laghu moves into the first, middle and the last positions in ya, ra and ta. ma and na represent all gurus and all laghus.

There are many features of Sanskrit prosody which distinguish it from the Greek. Greek prosody had its origin in music and dance, whereas in India, prosody began with the vedic chants. Also, whereas in Greek prosody, scanning is achieved though the analysis of the position and nature of disyllables, in Sanskrit, it is through the analysis of trisyllables and the two single syllables g and l.

We include at this point a few facts on vedic as well as classical prosody. In general, metrical music deals with three factors: the sound value of a syllable, syllabic quantity and the time taken for the utterence of a syllable. In vedic metres, the music depends only on the modulation of the voice in the pronunciation of the syllables; the essential features of the syllables, namely whether they are short or long do not matter. On the other hand, the music of classical metres depends on the essential features of the syllables, their variations and their order of succession. Hence, in classical prosody, a single letter could not be the unit of a metrical line as in vedic metres. A mere mention of the number of syllables which are all independent units sufficed to give an idea of the metrical line in the vedic metre and there was no need to give the essential features of the letters nor was it necessary to say how they were related to each other. But both these points required to be stated in the case of classical metres. Hence a method had to be found for scanning classical metres. Units of two syllables and their fourfold combinations are a choice and such a choice was indeed tried out by a Jain prosodist, as has been pointed out by H. D. Velankar in his book *Jayadāman*([13]). But these were found to be inadequate to express the basic constituents of the music of a metre, especially in the case of longer verses. So a new unit had to be found by the classical prosodists, which was neither too long nor too short. In ancient India, the number 3 was the least number which denoted multiplicity; the number 2 did not really signify plurality and indeed enjoyed too special a status. This is perhaps one of the reasons why, as H. D. Velankar suggests in his *Jayadāman*([13]), groups of three syllables were chosen by the ancient prosodists of India for scanning classical metres.

The second chapter of Piṅgala's *Chandas Sūtra* introduces and discusses various aspects of the seven basic vedic metres: *gāyatrī, uṣṇik, anuṣṭup, bṛhatī, paṅkti, triṣṭup* and *jagatī*. *Gāyatrī* consists generally of three *pāda*s of eight syllables each and hence has 24 syllables in all, and from then on, the number of syllables in these metres increases by 4 at a time, so that *uṣṇik* has 28 syllables, *anuṣṭup* has 32, *bṛhatī* has 36, *paṅkti* has 40, *triṣṭup* has 44 and *jagatī* has 48. Eight different varieties of these metres, are also discussed. Thus the seven basic vedic metres are divided into eight forms each, and totally there are 56 different kinds of metres.

In the third chapter, the notion of a *pāda* (foot) in Sanskrit prosody (which is very different from the notion of a 'foot' in Greek prosody) is discussed. Rules regarding filling of a *pāda* are also discussed. For example, in the *gāyatrī* when the number of syllables falls short of the required number of eight syllables, as in the following:

tatsaviturvareṇyam

तत्सवितुर्वरेण्यं

where there are only seven syllables, one should scan it as:

tatsaviturvareṇiyam

तत्सवितुर्वरेणियं

changing *y* to *iy*.

In this chapter, nine forms of the *gāyatrī* metre in terms of the number of *pāda*s are described. To give an example, one could have a *gāyatrī* stanza containing four *pāda*s with six syllables each, which is called *catuṣpāda gāyatrī*. Halāyudha, in his commentary gives such an example from the *Atharvaveda, Kāṇḍa 6, Sūkta 1.1* ([4]).

It is interesting to note that a non vedic *catuṣpāda gāyatrī* stanza (attributing it to the *Pāñcālā*s) is also quoted in the *Nidāna Sūtra (prapāṭhaka 1, Chando viciti)*([16]), whose meaning is unclear.

It is also interesting to note that in §16 of the *Ṛkprātiśākhya* ([20]), there is an example of such a stanza (stanza 7), which is given by Śaunaka. The stanza runs as follows:

indraḥ śacīpatir balena vīḷitaḥ
duścyavano vṛṣā samatsu sāsahiḥ

इन्द्रः शचीपतिर् बलेन वीळितः ।
दुश्च्यवनो वृषा समत्सु सासहिः ॥

This stanza is also found in the *Nidāna Sūtra (prapāṭhaka 1, Chando viciti)*([16]). A small part of this stanza occurs in the *Ṛgveda* (Eighth Maṇḍala, 19;20) namely:

yenā samatsu sāsahaḥ

येना समत्सु सासहः

(Yādava Prakāśa in his commentary of Piṅgala *Chandas Sūtra* notices this fact).

Similar forms of other metres are also discussed in this chapter. Mention is made of a class of those metres whose first and last verses have correct number of syllables, but whose middle verses have smaller number of syllables. Such metres are called *pipīlika madhya* that is, with a middle like that of an ant! For example, there are *gāyatrī* stanzas in which the first and the last *pāda*s have eight syllables but whose middle *pāda* has only three syllables. A general rule states that the number of syllables in the first *pāda* determines the metre.

To the seven basic metres are sometimes associated the seven *svara*s of music : namely *ṣaḍja, ṛṣabha, gāndhāra, madhyama, pañcama, dhaivata* and *niṣāda* (respectively); also the following colours: *sita* (silvery), *sāraṅga* (variegated), *piśaṅga* (brown), *kṛṣṇa* (black), *nīla* (blue), *lohita* (red) and *gaura* (white); and to the seven rishis: *Agniveśya, Kāśyapa,* *Gautama, Āngīrasa, Bhāradvāja,Kauśika* and *Vāsiṣṭha*. These identifications are intended as alternate methods to identify these metres, in case there is a confusion!

In chapter 4, after discussing fifteen kinds of vedic metres from *utkṛti* to *jagatī*, Piṅgala introduces the cryptic statement 'from now on classical metres' and from then on, he deals only with classical metres till the end of the book. He in fact deals in the rest of this chapter with the so called *mātrā vṛtta*s, that is those metres of classical Sanskrit based on the syllabic instants (a syllabic instant being the time taken to pronounce a short syllable: a long syllable takes twice as much time and is therefore said to constitute two syllabic instants). He discusses, in particular, the *Āryā* and the *Vaitālīya* metres. (We note, incidentally, that the *Āryabhaṭīya* of *Āryabhaṭa* is written in the *Āryā* metre.)

In the fifth chapter, Piṅgala discusses the so called *vṛtta chandas*. He classifies stanzas with four *pāda*s into three types: *sama, ardhasama* and *viṣama*. *Samavṛtta*s are those which consist of the same number of syllables in each *pāda*, while *ardhasamavṛtta*s have the same number of syllables in the first and the third *pāda*s, as well as in the second and the fourth *pāda*s. *viṣama vṛtta*s are those in which all the *pāda*s have unequal number syllables.

The aim of the sixth chapter of Piṅgala's *Chandas Sūtra* is principally to define the notion of *yati* (caesura). The *sūtra* which describes *yati* is *yati vicchedah*. The word *vicchedaḥ* signifies 'resting place'. It is the mechanical pause introduced in the middle of the verse. As against the irregular pauses in the vedic metres like *triṣṭup* and *jagatī*, it is regularly admitted in classical metres. While the origin of *yati* can be traced to the need for the ease of recitation, it evolved into an art and ornamentation in classical poetry. The concept of *yati* has been discussed at length by all the later prosodists and has become a regular feature of classical *vṛttas*. The effectiveness of *yati* in classical Sanskrit poetry, is best illustrated in the beautiful verses of the exquisite *Meghadūta* of Kālidāsa (in the slow-moving, majestic metre of *mandākrāntā*, a classical metre, with seventeen syllables, with pauses at the end of the fourth and tenth syllables in each *pāda*).

In the seventh chapter, Piṅgala describes and discusses metres *atijagatī, śakvarī, atiśakvarī, aṣṭi, atyaṣṭi, dhṛti, atidhṛti, kṛti, prakṛti, ākṛti, vikṛti, saṃkṛti, abhikṛti* and *utkṛti* which are the so called *atichan-*

das (hyper metres) containing 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92, 96, 100 and 104 syllables respectively. At the end of the chapter, he also explains the metre *daṇḍaka*.

The eighth chapter which is the concluding chapter of Piṅgala's book begins with the *sūtra, atrānuktaṃ gāthā*; Piṅgala's idea is to include in this chapter those metres which had not been mentioned in the earlier chapters.

The last fifteen *sūtra*s of this chapter (*sūtra*s 20 till 35) are the most interesting ones from the point of view of mathematics and deal with binary arithmetic and combinatorial questions arising out of the study of prosody. We shall discuss these in the next section.

§8. Piṅgala's sūtras and binary arithmetic

Since prosody deals with two symbols l and g and their repetitions, it is rather an easy matter for us (who live in this computer age) to guess[1] that the study of prosody should naturally lead to questions on binary arithmetic. Indeed, the study of prosody did lead the ancient Indian mathematicians to binary arithmetic and combinatorics, as is evidenced by the *sūtra*s 20-35 in the eighth chapter. As is usual with Piṅgala, these *sūtra*s are cryptic to the point of being obscure. However, as is customary with the ancient Indian system of preserving knowledge, the later commentators of Piṅgala's *Chandas sūtra* have provided ample explanations of the *sūtra*s ([17], [22]).

The *sūtra*s 20-23 deal with the construction of the so called *prastāra* of a metre, which can be translated roughly into English as a matrix or an array of syllables. The *laghu*s and *guru*s in a metre of a given length are listed horizontally as rows (or lines) in a *prastāra*. This device of a *prastāra* can be thought of as a table written either on the ground or on a board. The rules for the construction of a *prastāra*, for metres of length one, two or three are given in these *sūtra*s. For example, the *prastāra* for a metre of length 1 is obtained by first writing the symbol g (for *guru*) and beneath it the symbol l (for *laghu*). The *prastāra* for

[1]One remembers the words of Schiaparelli, the Italian historian of Early Greek Astronomy who wrote in the introductory section of his paper on the work of Eudoxus on Astronomy: "Tutto il nostro merito sta nell'esser venuti al mondo piu tardi": 'Our sole merit consists in having come to the world a little later'.

a metre of length 2 starts with a horizontal row with two *gurus*: $g \quad g$. We begin the next row , by writing l (for a *laghu*) below the first entry g of the first row and write g below the second entry g of the first row, so that this row reads $l \quad g$. In the third row, we begin with a g and write a l beneath the next entry g of the second row so that, the third row reads $g \quad l$. We begin the fourth row with an l and write a l below the the next entry l of the third row. The *prastāra* for a metre with two syllables is now complete and is the array of four horizontal rows

g	g	1
l	g	2
g	l	3
l	l	4

consisting of two syllables each. The general rules for constructing the *prastāras* of metres of a given length n are similar and explained by the *sūtras*. Namely, we start with a horizontal row consisting entirely of n *gurus*. The rest of the rows of the *prastāra* are constructed by using the following rule: Start any row and continue filling the row with *gurus until* we see for the *first* time a *guru* in the previous row. Then write a *laghu* as the entry for this row below this *guru* and from then on, copy the rest of the syllables from the previous row. We continue filling rows this way until we reach a row consisting of all *laghus*, where we stop. [2] This method applied to two syllables gives obviously the *prastāra* of two syllables we have written above. The *prastāra* for a metre of three syllables, using the rule described above gives the table for the eight *gaṇas* (trisyllables) we wrote down in the beginning of the previous section.

The *sūtra* 23 reads *vasavastrikaḥ*, which simply enumerates the number of trisyllables as eight! (there are eight *vasus* according to the vedic lore!)

[2]As has been kindly pointed out by Professor M.G. Nadkarni, this rule applied to infinite sequences of zeros and ones ($g = 0$, $l = 1$) gives rise to a transformation on the space of sequences of zeros and ones. It is a very basic object in ergodic theory called dyadic coding machine or odometer transformation, and when viewed as a transformation of the unit interval, it is called von Neumann transformation, a name given by Kakutani. This transformation plays a very important role in orbit-equivalance theory and related areas.

Let us also add one more fact regarding the construction of the *prastāra*. We number the rows of a *prastāra* serially with the first row of the *prastāra* consisting of all *guru*s being numbered as 1.

Before discussing the rest of the *sūtra*s, it is perhaps convenient to introduce a stanza which lists the various techniques, termed as *pratyaya*s, by which some arithmetic questions related to metres can be analysed. This stanza is found in text books on prosody subsequent to Piṅgala's work. For instance, it is found in Kedāra Bhaṭṭa's *vṛtta ratnākara*, ([25]) Yādava Prakāśa's commentary of Piṅgala's *Chandas Sūtra*, Hemacandra's *chandonuśāsana*, and in many other works on prosody. The stanza in question ([25], p. 187) reads as follows:

prastāro naṣṭamuddiṣṭam ekadvayādi lagakriyā
saṅkhyā caivādhvayogaśca ṣaḍete pratyayāḥ smṛtāḥ

प्रस्तारो नष्टमुद्दिष्टं एकद्वयादि लगक्रिया ।
सङ्ख्या चैवाध्वयोगश्च षडेते प्रत्ययाः स्मृताः ॥

As we said, the above stanza enumerates the various components of some of the arithmetic aspects of prosody, namely: (*i*) *prastāra* (whose meaning we just now explained), (*ii*) *naṣṭam*, (*iii*) *uddiṣṭam*, (*iv*) *ekadvayādilagakriyā*, (*v*) *saṅkhyā*, (*vi*) *adhva yoga*.

The following stanza ([25], p. 188) summarises what we said already about the way a *prastāra* is constructed:

pāde sarvagurāvādyāllaghuṃ nyasya guroradhaḥ
yathopari tathāśeṣaṃ bhūyaḥ kuryādamuṃ vidhim
ūne dadyād gurūneva yāvat sarvalaghurbhavet
prastāro'yaṃ samākhyātaḥ chandovicitivedibhiḥ

पादे सर्वगुरावाद्याल्लघुं न्यस्य गुरोरधः ।
यथोपरि तथाशेषं भूयः कुर्यादमुं विधिम् ॥
ऊने दद्याद् गुरूनेव यावत्सर्वलघुर्भवेत् ।
प्रस्तारोऽयं समाख्यातः छन्दोविचितिवेदिभिः॥

We shall describe now each of the other aspects listed above related to the arithmetic of prosody. *Sūtra*s 24 and 25 of chapter eight of Piṅgala's *Chandas Sūtra*, which read:

(24) l'ardhe
and
(25) saike g

refer to the process of *naṣṭam* and this word means 'vanishing' or 'disappearance'. Suppose that the *prastāra* of the metre (which is usually written on the sand) has been erased by mistake. The process described shows how to recover the metre only through the knowledge of the number of the row in which the particular metre had appeared. This process is illustrated by the following example: Suppose that we know that a certain metre with a fixed number of syllables say 6, occurs as the 44th row in the *prastāra*, how does one write down the corresponding metre? The answer is given by the two *sūtra*s above as elaborated further by the following stanza in *Vṛtta ratnākara* ([25], p.192), which explains the process *naṣṭam*. (There are similar explanations of these *sūtra*s in Halāyudha's and Yādava Prakāśa's commentaries.)

> *naṣṭasya yo bhavedaṅkastasyārdhe'rdhe same ca laḥ*
> *viṣame caikamādāya tasyārdhe'rdhe gururbhavet*

नष्टस्य यो भवेदङ्कस्तस्यार्धेऽर्धे समे च लः ।
विषमे चैकमाधाय तस्यार्धेऽर्धे गुरुर्भवेत् ॥

The procedure indicated is best explained by applying it to the example mentioned above: the number 44 being divisible by 2, we write an l (for *laghu*) and divide 44 by 2 to get 22. The number 22 being still divisible by 2, we append an l to the earlier *laghu* and divide 22 by 2, to get 11. Up to this point, the procedure is indicated by *sūtra* 24 which says *if it is possible to halve, then an* l. When we, however, hit the number 11 which is odd, *sūtra* 25 takes over and it says *otherwise add* 1 *and a* g. Now add 1 to 11 and write a g (a *guru*). The number now is 12, which is divisible by 2; and we divide by 2 to get 6. Now 6 being divisible by 2, *sūtra* 24 applies, we write an l and divide 6 by 2. We get 3 which is odd. *Sūtra* 25 tells us that we should add 1 and write a g. We now get 4, which we divide by 2 to get 2. We write an l and divide 2 by 2 to get 1 as quotient and we stop here (since the metre has 6 syllables) and the metre we are looking for is

$$l \quad l \quad g \quad l \quad g \quad l$$

This is also the general rule given in the *sūtra*s 24 and 25 (and explained in the stanza) for writing down a metre, given the number of its row in the *prastāra*.

The process *uddiṣṭam* is indicated by two *sūtra*s of the Chapter 8 of Piṅgala's *Chandas Sūtra* which read:

(26) pratilomagaṇam dvirlādyam
and
(27) tatogyekaṃ jahyāt

and expanded upon in the following couplet of Kedāra Bhaṭṭa ([25], p.194):

uddiṣṭaṃ dviguṇādyadyuparyaṅkānsamālikhet
laghusthā ye tu tatrāṅkāstaiḥ saikairmiśritairbhavet

उद्दिष्टं द्विगुणाद्याद्युपर्यङ्कान्समालिखेत् ।
लघुस्था ये तु तत्राङ्कास्तैः सैकैर्मिश्रितैर्भवेत् ।

These *sūtra*s, as interpreted by the couplet above, answer the following question: Suppose that one is given a metre with a certain number of syllables what is the number of the row representing this metre in the *prastāra*?

The process *uddiṣṭam* is thus the converse of *naṣṭam* and can be translated as 'determination'; it gives a method of determining the number of the row representing a metre with a certain number of syllables.

The answer, as given by the couplet is the following: We make the number 1 correspond to the first syllable from the left and from then on, make powers of 2, namely $2, 4, 8, \ldots$ correspond to each succeeding syllable. Ignoring the powers of 2 corresponding to the *guru*s of the metre and adding the powers of 2 corresponding only to the *laghu*s of the metre and increasing this sum by 1 gives the requisite number of the line in the *prastāra*. (Put in the mathematical language, one thinks of the metre as a mnemonic for a dyadic expansion by thinking of the *laghu*s as representing 1 and the *guru*s as representing 0!) Let us consider as an example, the metre

$$l \quad l \quad g \quad l \quad g \quad l,$$

as above. Then the number in question is $1+2+8+32=43$ increased by 1, that is 44.

It should be remarked at this point that the *sūtras* 26 and 27 as stated by Piṅgala *do not* suggest the above procedure outlined by the couplet. The *sūtras* themselves have been interpreted by Halāyudha in a different way and this interpretation is also found in the commentary of Piṅgala's *Chandas sūtra* by Yādava Prakāśa. We shall discuss this presently. But before doing this, let us note that the processes of *uddiṣṭam* and *naṣṭam* described above, together give a one to one correspondence between non-negative integers and their dyadic expansions, via, metres. In fact, given any metre we get an integer by the process described above, (by assigning the value 1 to a *laghu* and 0 to a *guru* and assigning the value 2^{i-1} to the syllable which occurs at the ith position from the left; summing these numbers and adding 1 to it we get the number of the row corresponding to this metre in the *prastāra*). Conversely, *sūtras* 24 and 25 (explained further by the process of *naṣṭam*) assign to every integer a metre. These two processes are obviously inverses of each other. We further note that the metre which consists only of *gurus* corresponds to the dyadic expansion of 0 and since this is the first row of the *prastāra*, the number of the row corresponding to any metre is one more than the number given by the corresponding dyadic expansion.

We shall now give the interpretations of Halāyudha and Yādava Prakāśa of the *sūtras* 26 and 27 of Piṅgala, which give a very interesting method of computing the number represented by a dyadic expansion.

We shall explain this principle now, mainly through examples, and then state the general principle without proof (the proof is easy to establish).

Consider for example the string of syllables:

$$l \quad g \quad l$$

The rule given by the *sūtras* (and explained by Halāyudha and Yādava Prakāśa in their commentaries), applied to the above metre says that we first look at the syllable on the extreme right. Noting that this syllable is a *laghu*, we attach the value 2 to it. We next look at the syllable to its immediate left. Noting that this is a *guru*, we attach to it the value 3 which is *one less than twice* the number 2, attached to the previous syllable. We then look at the next syllable to the left which is a *laghu*. To this we attach twice the value attached to the previous syllable and this is 6. The number of the row representing this metre in the *prastāra* is 6!

We note that according to our earlier computation, the above row of syllables represents the number $1 + 4$ increased by 1 which is 6 again!

As we shall remark presently, the above process applies, in general, to all metres of a given length n and the number associated to the first syllable is indeed the number of the row of the given metre in the *prastāra* of metres of length n.

We look at the *gāyatrī* metre, considered earlier, as another example.

$$l \quad l \quad g \quad l \quad g \quad l$$

We assign the value 2 for the *laghu* on the extreme right, the value $4 - 1 = 3$, for the next syllable on its left which is a *guru* and then 6 for the next which is a *laghu*, then 11 for the next syllable which is a *guru* and 22 for the next syllable which is a *laghu* and finally 44 for the first syllable on the extreme left which is a *laghu*. This is the number for the *gāyatrī* row in the *prastāra* for a metre of six syllables!

The general rule can now be formulated: If we take a metre of any length, and wish to find out what its number is as a row in the *prastāra* of metres of this length, we start by giving the value 2 or $2-1 = 1$ to the syllable on the extreme right, according as it is a *laghu* or a *guru*. We multiply this number by 2 and attach this number to the next syllable on its left, if it happens to be a *laghu* or attach this number decreased by 1 if this syllable happens to be a *guru*. Keep on repeating this procedure till we reach the beginning syllable of the metre. The number attached to this syllable is the number of the row in the *prastāra*.

It is easily verified that the number obtained by the procedure indicated above coincides with the number given by the dyadic expansion (by assigning the value 1 to a *laghu*, 0 to a *guru*, *increased by 1*).

Thus, the above is another method of finding a number through its dyadic expansion and this does not use *addition* of terms (as the earlier one did) and is more algorithmic, suited to the computer. In this sense, this ancient method is as modern as that of the computer! Actually, the *sūtra* 26 says that we first reverse the metre and carry the process from left to right.

We now turn to the *sūtra*s 28 to 32 and 34, 35 of Piṅgala, which deal with the combinatorics given rise to by the study of metres. The *sūtra*s in question are:

(28) dvirardhe;

(29) rūpe śūnyam;

(30) dvi śūnye;

(31) tāvadardhe tadguṇitam;

(32) dvirdyūnaṃ tadantānām;

(34) pare pūrṇam;
and

(35) parepūrṇamiti.

The questions asked and answered are: *How many metres with a given length have gurus ocurring once, twice etc? How many metres are there with a given length?* These questions which naturally arise in the study of prosody, obviously deal with the theory of permutations and combinations. We shall see that in this connection, the so called *Pascal triangle*, from which one can read off the binomial coefficients was already constructed by the ancient prosodists of India.

These topics are covered under the headings *ekadvayādilagakriyā* and *saṅkhyā* by the later prosodists like Kedāra Bhaṭṭa and others. (As a matter of fact, Piṅgala's *Chandas Sūtra* deals with these topics in the reverse order.) Piṅgala's *sūtra*s 28-32 treat *saṅkhyā* and 34 and 35 with the computation of number of metres of a given length with prescribed number of *guru*s and *laghu*s in it, through the combinatorics of what is now known as the *Pascal triangle*.) The two verses in Kedāra Bhaṭṭa's work ([25], p.196) which describe the first process is the following:

varṇān vṛttabhavān saikān auttarādharya taḥ sthitān
ekādikramataścaitānuparyupari nikṣipet
upāntyato nivarteta tyajennekaikamūrdhvataḥ
uparyādyāt gurorevamekadvayādilagakriyā

वर्णान् वृत्तभवान् सैकान् औत्तराधर्यतः स्थितान् ।
एकादिक्रमतश्चैतानुपर्युपरि निक्षिपेत् ॥

उपान्त्यतो निवर्तेत त्यजेन्नेकैकमूर्ध्वतः ।
उपर्याद्यात् गुरोरेवमेकद्व्यादिलगक्रिया ॥

The method to find the number of metres of length n in which *guru*s and *laghu*s occur once, twice etc, as suggested in the above verse, is the following:

We start with a row of length $n + 1$ consisting of the number 1. (In what follows we assume for simplicity that $n = 6$ and the next figure illustrates the procedure for $n = 6$.)

1	1	1	1	1	1	1
1	2	3	4	5	6	
1	3	6	10	15		
1	4	10	20			
1	5	15				
1	6					
1						

We start the second row with a 1. For the next position we take the sum of the number which precedes it in the row (which is 1 in our example) and the number of the previous row in the position above it (which is 1 again in our example), and the sum here is 2. We choose the next number of the row to be once again the sum of the two numbers, one which is in the preceding position in the row and the number in the position above it in the previous row.

Hence, in this case, we take 2+1=3 as the next number in the second row. The third number in the second row is chosen similarly and we continue this procedure, and *end* the second row with the number of entries one less than that of the first row. In our example, the second row has therefore 6 entries, the last entry being $5 + 1 = 6$. We start the third row once again with a 1; choose the number for the second position of the row the sum of the number in the row in the position preceding it which is 1 in our case and the number in the position above it in the second row which is 2 so that we take 1+2=3 as the second number of the third row. We stop this row once again with the number of its entries one less than the second row, which is five in our example, the last entry being 10+5=15. We continue this process until we stop with the $(n+1)$th row which has just one entry namely 1.

The number of metres with n syllables in which *guru* appears only once is given by the last number of the second row, which is n. This number is obviously also the number of metres of length n, in which the *laghu*s appear $n-1$ times. The number of metres of length n in which *guru* appears exactly twice is given by the last number of the third row which is seen to be $n(n-1)/2$. More generally, the number of metres of length n in which the *guru* appears i-times is given by the last term of the $(i+1)$th row, and which is $\binom{n}{i}$.

Thus, the array constructed with the specifications of the two verses above gives a computation for the binomial coefficients and is the so called *Pascal triangle*, (with its base tilted by 45 degrees) which was constructed by Pascal in 1654. This device had however been used by the Indian prosodists, under the name *meru prastāra*, at least two thousand years earlier, in connection with the study of metres.

It is interesting to note that Bhāskarācārya II, the mathematician, who lived in the 12th century A.D, in his famous book of problems called *Līlāvatī*, has the following verse ([8]) which asks for the number of metres with a prescribed length and with a specified number of *guru*s or *laghu*s (and the commentary provides a very simple algorithm for finding these.) The verse in question (for the *gāyatrī* metre), for example, is the following:

prastāre mitra gayatryāḥ syuḥ pāde vyaktayaḥ kati
ekādi guravaścāśu kathyatāṃ tatpṛthak pṛthak?

प्रस्तारे मित्र ! गायत्र्याः स्युः पादे व्यक्तयः कति ।
एकादिगुरवश्चाशु कथ्यतां तत्पृथक् पृथक् ?

The figure below gives the solution: We begin a row with the length of the metre as its first entry. The succeeding entries of the row are those gotten by decreasing this number successively by one at a time, the last entry of the row being 1. Below this row, we start a new row beginning with 1, the succeeding numbers in this row being those obtained by increasing the numbers successively by one, the last entry of this row being the length of the metre. We fill in a new row above these two rows by the following numbers. The first entry in the new row shall be the number obtained by multiplying the first entries of the two rows below, so that we get as the first entry of the row above as $1 \times 6 = 6$. The next entry of the new row is obtained by multiplying the first two entries of the first row and dividing it by the product of the first two entries of the second row, so that we get in our example, the number in the new row to be $\frac{6\times5}{1\times2} = 15$. The third entry in the new row shall be the product of the first three entries of the first row divided by the product of the first three entries of the second row, which for our example is the number $\frac{6\times5\times4}{1\times2\times3} = 20$.

We continue the process, till we get the last entry of the new row which is 1.

6	15	20	15	6	1
6	5	4	3	2	1
1	2	3	4	5	6

More generally, for any metre with length n, we get, as the first entry of the new row, the number $n = n \cdot 1 = \binom{n}{1}$, the second entry to be $\frac{n\cdot(n-1)}{1\cdot2} = \binom{n}{2}$ and, more generally, for the ith entry the number $\frac{n\cdot(n-1)\cdots(n-i+1)}{1\cdot2\cdots i} = \binom{n}{i}$.

These, as we know, give the number of metres of length n, in which the *guru* (and similarly the *laghu*) occurs exactly once, twice, $\ldots, i$-times. In particular,in the example of the *gāyatrī* metre, the numbers are $6, 15, 20, 15, 15, 6$.

Līlāvatī ([8], Appendix p. 48) has another problem on the determination of the number of *sama, ardhasama* and *viṣama vṛtta*s in the metre *anuṣṭbh*, which is preceded by a general rule valid for any metre, given below.

pādākṣaramitagacche guṇavargaphalañjaye dviguṇe

samavṛttānāṃ saṅkhyā tadvargo vargavargaśca
svasvapadonau syātāmardhasamānāñca viṣamāṇām

पादाक्षरमितगच्छे गुणवर्गफलञ्चये द्विगुणे ।
समवृत्तानां सङ्ख्या तद्वर्गो वर्गवर्गश्च ।
स्वस्वपदोनौ स्यातामर्धसमानाञ्च विषमाणाम् ॥

The number of syllables in the four verses of the *vṛtta* in anuṣṭup, being 32 (the anuṣṭup, has 8 syllables), the number of arrangements of the long and short syllables of all the *pāda*s is $2^{32} = 4294967296$. Evidently, the number of arrangements where the *pāda*s are all alike is the number of arrangements of the syllables in a single *pāda* and is hence $2^8 = 256$. The number of arrangements occuring as *ardhasama vṛtta*s is $2^{16} - 256 = 65280$. The number of *viṣama vṛtta*s which is the total number of all the arrangements 'minus' the number of possibilities where two *pāda*s are alike is $2^{32} - 2^{16} = 4294967296 - 65536 = 4294901760$.

We discuss finally *saṅkhyā*. As we remarked earlier, this has been discussed in the Piṅgala's *sūtra*s 28-32. Kedāra Bhaṭṭa ([25], p.201), on the other hand, has the following verse describing *saṅkhyā*.

lagakriyāṅkasandohe bhavetsaṅkhyā vimiśrite
uddiṣṭāṅkasamāhāraḥ saiko vā janayedimām

लगक्रियाङ्कसन्दोहे भवेत्संख्या विमिश्रिते ।
उद्दिष्टाङ्कसमाहारः सैको वा जनयेदिमाम् ॥

As the verse says, there are two methods of computing the number of metres of length n. One can either sum up the numbers (obtained by the process of *lagakriyā*) which count the number of metres in which the *guru*s occur once, twice, etc. (These numbers are the entries on the extreme right in the *meru prastāra* we constructed earlier.) In other words, one is here summing up all the binomial coefficients of n and therefore one gets $\binom{n}{0} + \binom{n}{1} \cdots + \binom{n}{n} = 2^n$, which is obviously the number of metres of length n.

Otherwise, one can use the process of *uddiṣṭam*. We note that the required number is the number of metres in the *prastāra* of the metre.

The length of the metre being n, as we have remarked earlier, the last row of the *prastāra* consists of n *laghu*s, and then we know that it corresponds to the dyadic expansion $1 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$, and the number of rows of the *prastāra* is gotten by adding 1 to it, so that one obtains 2^n.

Piṅgala's *sūtra*s, mentioned above, give a somewhat elaborate method of arriving at this number, which we shall not discuss, since, in any case, it is simple combinatorics.

The *sūtra* 33 of Piṅgala reads *ekona adhvā*, which deals with the space required on the sacrificial ground for writing the *prastāra* of a metre of a given length. Since there is no mathematics involved in it, we shall pass over this *sūtra* and its explanation given by Kedāra Bhaṭṭa in his *Vṛtta Ratnākara*.

§9. Concluding remarks

To summarise,our aim in this article has been to highlight the contributions of Piṅgala to Sanskrit prosody with a special emphasis on the combinatorial aspects. As we mentioned earlier, the influence of Piṅgala on the later prosodists has been profound. Particularly interesting is the development of Prakrit prosody with emphasis on *mātra vṛtta*s. One of the greatest of the later prosodists is Hemacandra whose name we already have mentioned. The construction of the *prastāra* and the other devices mentioned in the earlier section can be extended to *mātra vṛtta*s too, as has been explained by Kedāra Bhaṭṭa [25]. The work of Hemacandra ([8], [2]) has a complete chapter on the combinatorics of prosody with special reference to *mātra vṛtta*s (in particular, to the *Ārya* metre). We do not discuss these here.

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Shedding Some Localic and Linguistic Light on the Tetralemma Conundrums

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Abstract

Numerous authors over the centuries have puzzled over what has been called the Buddhist paradigm of catuṣkoṭi. A classic example: the four statements, considered both mutually exclusive and jointly exhaustive,

(i) the Tathāgata exists after death;

(ii) the Tathāgata does not exist after death;

(iii) the Tathāgata both does and does not exist after death;

(iv) the Tathāgata neither does nor does not exist after death.

We offer some linguistic *gedanken*-experiments illustrating everyday situations in which appropriate analogues to the above four statement-forms are entirely plausible as mutually exclusive or jointly exhaustive alternatives; and we offer a framework, based on the logical paradigms of locale or topos theory, illustrating how forms (iii) and (iv), in particular, need be neither contradictory, nor paradoxical, nor even mutually equivalent.

1 Foreword

As an exemplary model – what in German might be called a *Vorbild* or *Musterexemplar* – of catuṣkoṭi or tetralemma, one would be hard-pressed to find a more quintessentially perfect instance than the following, taken from verse XVIII.8 of the *Mūlamādhyamakakārikā* by Nāgārjuna, as kindly rendered into English by the generous referee:

> Anything is either true,
> Or not true,
> Or both true and not true,
> Or neither . . . .

This fragment provides what a mathematician of a certain bent might call a *universal example* of catuṣkoṭi in all regards – form and content, structure and message – without a single superfluous word or restriction.

Of course, to the reader steeped in the bivalent logical tradition prevalent in the West from the time of Aristotle, or even before, through the time of Boole, and beyond, already the first two lines of this fragment would seem to cover all the bases, with the last two being quite superfluous, little more than incomprehensible, contradictory, mystifying mumbo-jumbo, if not downright misleading mystical nonsense.

The very modest aim of the presentation that follows will be to tease out of the catuṣkoṭi any lingering fiber of paradox, conundrum, or mysticality, so as to allow such a reader to recognize, in each of the four alternatives that the catuṣkoṭi sets forth, a familiar, viable, and relevant state of affairs. The means by which to accomplish this aim will number but two: suitable models of (nonstandard) logical systems (cf. [R] for a thorough if technical introduction to such ideas), and examples from everyday language.

The reader expecting anything more will, alas, come away disappointed. Neither the history of the catuṣkoṭi, nor the role of catuṣkoṭi in Indian philosophy, nor any critical analysis of their many commentators, nor any sensitive comparisons or contrasts of those commentators, or their comments, one with another – no such scholarly discourse – will be found here. For such material, the reader is better advised to visit the pages of [B], [C], [G], or [Si], or, even better, the many works cited therein.

Nor will the reader find any attempt to provide information as to the nature of Truth, or Reality, or what it means to be Valid, or to Exist – here again, for etymological as well as for epistemological enlightenment, our advice would be to consult articles like [K] and [Sö], or to browse through the pages of [JIPR] and of its subsequent sister volumes, where similar articles have appeared.

Finally, beyond offering all due gratitude to the reader who can forgive these omissions (and abject apologies to the reader who cannot), I must express profound thanks to Professors Emch and Sridharan for having encouraged the preparation of the present material in the first place; to the referee (alas, anonymous), whose numerous valuable suggestions upon an earlier draft, I hope, I have adequately incorporated into the present revision; to Professor P. Vanchinathan for a masterful translation of my HTML submission into the requisite LaTeX; to the airlines UAL and Lufthansa, whose frequent flyer program and aircraft, respectively, graciously facilitated my participation in the Bangalore conference by providing complimentary air transport between the North American continent and India; and to the Faculty Research Grant program of Wesleyan University, for its generosity in underwriting selected additional travel expenses connected with the presentation of this material at Bangalore.

2 Taming the Terrible Catuṣkoṭi

What the extract from Nāgārjuna cited in the foreword suggests, as regards the catuṣkoṭi quoted in the abstract (to be found as item /1/ on page 28 of [B]), is that, writing P for the proposition that the Tathāgata exists after death, the four propositions

(i′) P

(ii′) $\neg P$ (not-P)

(iii′) $P \& \neg P$ (P and not-P)

(iv′) $\neg P \& \neg\neg P$ (neither P nor not-P)

(corresponding to (i)–(iv)) are mutually exclusive and cover all possibilities. What sort of logic can be at work here?

Classically, of course, at least in the western tradition, where P and $\neg P$ are complementary and $\neg\neg P = P$, the last two formulations are identically trivial and the first two already cover all possibilities (principle of excluded middle, or *tertium non datur*).

In the intuitionistic logic of a topos, on the other hand – and, unlike in an earlier lecture [L], we shall refrain here from attempting to offer any introduction to the notion of topos, or to the sort of logic prevailing there, preferring to send the interested reader to such standard expositions of those matters as [J] (especially Chapter 5, sections 1 and 2), or [L&S] (especially the marvellously informal overview of pp. 123–128), or [M&M] – the first two formulations are no longer complementary. They do remain mutually exclusive, however, and the last two, consequently, are still identically trivial. It is just that the first two need no longer cover all possibilities, that is, the principle of excluded middle need no longer hold (concrete illustrations of such state of affairs appears below).

If, instead, we envision a logic dual to that of a topos, more like the logic of the lattice of closed subsets of a topological space, we finally reach a situation where both $P\&\neg P$ and $\neg P\&\neg\neg P$ may be non-trivial. But now $P, \neg P, P\&\neg P$, and $\neg P\&\neg\neg P$ may well no longer be mutually exclusive. Indeed, at least for closed P, we have the order-inclusions

$$\begin{aligned} \neg P\&\neg\neg P &\leq P\&\neg P \leq P \qquad \text{and} \\ \neg P\&\neg\neg P &\leq P\&\neg P \leq \neg P, \end{aligned}$$

so that if P is "regular-closed", say, that is, if $P = \neg\neg P$, the last two formulations coincide and fall within both P and $\neg P$ (indeed, they constitute the boundary of P).

Somehow, $\neg P$ must not be getting treated purely as the negation of P. Let us write Q temporarily for this negation of P, and see what we can make of statements (i) through (iv) in such a setting. They become

(i″) P,

(ii″) Q,

(iii″) $P\&Q$, and

(iv″) $\neg P\&\neg Q(= \neg(P \vee Q))$,

where the last formulation is logically tantamount to the negation of "P or Q (or both)", i.e., to the negation of what the first two alone jointly cover. Certainly the last item here excludes each of the earlier ones, and all are, in general, non-trivial. But if all four are to be mutually exclusive, what (i$''$) and (ii$''$) are implicitly intending to represent must surely be rather

(i$'''$) $P\&\neg Q$ and

(ii$'''$) $\neg P\&Q$,

respectively. Then, at least classically, we obtain the four mutually exclusive, jointly exhaustive, atomic generators of the free Boolean algebra on the two free generators P and Q, viz.:

(a) $P\&\neg Q = P - Q$,

(b) $\neg P\&Q = Q - P$.

(c) $P\&Q = P\&Q$, and

(d) $\neg P\&\neg Q(= \neg((P - Q) \vee (Q - P) \vee (P\&Q)) = \neg(P \vee Q))$.

But how is one now to make any sense of the hope that Q may stand for $\neg P$? That is, how shall we maintain the mutual exclusivity and individual non-triviality of the four items

(a$'$) $P\&\neg\neg P$,

(b$'$) $\neg P\&\neg P$,

(c$'$) $P\&\neg P$, and

(d$'$) $\neg P\&\neg\neg P$,

obtained from (a)–(d) by putting $\neg P$ in place of Q ?

Let us simplify, for the moment, by assuming that $\neg\neg P = P$, so that the four conjunctions above become

(a$''$) $P\&P$,

(b$''$) $\neg P\&\neg P$,

(c$''$) $P\&\neg P$, and

(d$''$) $\neg P\&P$.

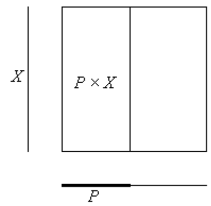


Figure 1

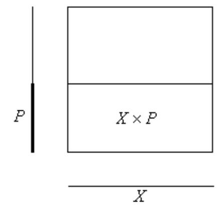


Figure 2

Next, let us imagine that the second occurence of P in each of these four conjunctions is merely a homonym for the P that occurs first. Mostly, in living languages, homonyms are words that sound alike, but are spelled differently and have different meanings, like *red*, the color, and *read*, the past participle, or *pear*, the fruit, *pair*, the duo or couple, and *pare*, the verb meaning to peel (and perhaps also cut up) a fruit (perhaps even a pear) or vegetable. But there are homonyms also with both sound and spelling identical, like *sucker*, which can at once signify a person easily duped or taken advantage of, or a tendril on a vine.

How may we realize the two occurrences of P in (a'')–(d'') as mere homonyms of each other? It would be enough, for example, were our lattice of propositions somehow spatial, that is, representable as some sort of subsets of some particular space X, to place ourselves in the Cartesian product $X \times X$ of the space X with itself. For now, corresponding to P, there arise two clearly distinguishable homonyms of P in $X \times X$: one, the "vertical cylinder" $P \times X$ over the P in the first spatial factor X; the other, the "horizontal cylinder" $X \times P$ alongside the P in the second factor X (cf. Figures 1 & 2).

If we now simply treat each first occurrence of P in the forms (a'')–(d'') as instances of the vertical cylinder $P \times X$, and each second occurrence as the horizontal one $X \times P$, then our four conjunctions correspond to the four rectangles in Figure 3 in the following page ($P \& P$ being interpreted, for example, as the intersection, $P \times P$, of $P \times X$ with $X \times P$, etc.).

For what it is worth, we exhibit a topos whose internal logic has system of truth values inherently of this form. Indeed, where $\mathbf{S}$ is any of the very classical topoi of absolutely standard sets – say, made up of the sets in Gödel's constructive hierarchy – the topos $\mathbf{S} \times \mathbf{S}$ of ordered

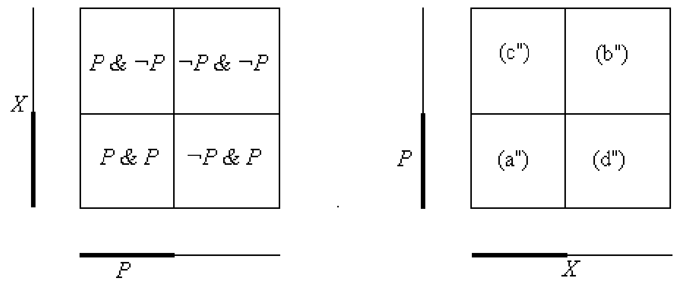


Figure 3

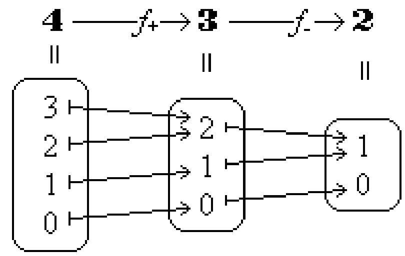
pairs of such sets is such a topos. Its truth value object is the ordered pair $(\mathbf{2}, \mathbf{2})$ consisting of two copies of the usual two-element Boolean algebra formed from the ordinal number $\mathbf{2} = \{0, 1\}$, and this has exactly four global elements: (1, 1) and (0, 0), playing the roles of True and NotTrue, and serving as counterparts of $P \& P$ and $\neg P \& \neg P$, respectively; and (1, 0) and (0, 1), playing the roles of BothTrueAndNotTrue and NeitherTrueNorNotTrue, counterparts in turn of $P \& \neg P$ and $\neg P \& P$.

Not every topos whose truth value object has exactly four global elements has them arranged quite in this way, however. For example, if we topologize the ordinal number **3** (whose points are the smaller ordinals 0, 1 and **2**) by declaring open exactly those subsets of **3** that happen themselves to be ordinal numbers (viz., $\emptyset$ (the empty subset), $\{0\}, \{0, 1\}$ and all of **3**), then the topos of *sheaves* on this space **3** has truth value object whose global elements likewise number four, but correspond exactly, even as to their ordering, to the four open subsets of **3** that make up the topology just described. Here, between True and NotTrue (or **3** and the empty set) we have two intermediate truth values, each neither True nor NotTrue, but one "more true", as it were, and "less not true", than the other:

$$\mathsf{NotTrue} = \emptyset < \{0\} < \{0, 1\} < \mathbf{3} = \mathsf{True}.$$

To within isomorphism, this topos may also be depicted as the topos of double-transitions among sets, that is, as configurations of the form

$$A \xrightarrow{f} B \xrightarrow{g} C$$



made up of three sets and two functions, as depicted. The truth value object for this topos is the configuration above, where **4, 3**, and **2** are the ordinal numbers $\mathbf{4} = \{0, 1, 2, 3\}, \mathbf{3} = \{0, 1, 2\}$, and $\mathbf{2} = \{0, 1\}$, and the functions f_+ and f_- both carry 0 to 0 and 1 to 1, but $f_-(2) = 1$, while $f_+(2) = f_+(3) = 2$, as depicted above.

The four global elements here are simply the four length-two paths, or orbits, seen to emanate from the various members of **4**, the uppermost and lowermost of which it seems plausible to accept as playing the roles of True and NotTrue, respectively, while the remaining two paths, clearly neither True nor NotTrue, somehow represent the values "more True than NotTrue" and "more NotTrue than True". Or perhaps the catuṣkoṭian expressions "both true and yet not true" and "neither true nor yet not true" better convey the sense of these intermediate truth values, though we suspect this is not an illustration of the classical paradigm the catuṣkoṭi had in mind.

But in fact, the logic of this topos does realize the way apparent contradictions are commonly used in everyday speech. A daiquiri made with far too much lime juice, for example, and a little too much sugar, may well be called both sweet and not sweet; a coffee prepared with just barely not enough sugar for the taste of a particular coffee-drinker may be disparaged as neither sweet nor not sweet. If the best student to pass through your department in the past ten years has an uncanny knack for getting arrested at student political demonstrations, you will be apt to wonder whether your department should once again post bail for this student who is both really very bright and yet not really very bright. Or, of another student, not quite so bright – generally dealing very well with the more difficult problems and readings, but sometimes inexplicably failing utterly when faced with far simpler ones – and yet having an investment acumen that is simply uncanny, you may well think, somewhat perplexedly, this student is neither really all that bright, nor not really all that bright.

There are, of course, also everyday linguistic settings in which the last two catuṣkoṭi options (iii) and (iv), far from being mutually exclusive, coincide completely. This state of affairs corresponds, perhaps, to the Trairāśika viewpoint (cf. [B], p. 35). A grape-fruit, for example, sour, to some extent, like all its kin, but remarkably less so than most, you might be equally happy to describe as both sour and not sour, or as neither sour nor not sour. Would you like a topos whose truth value object epitomizes just this situation, *not* envisioned in the catuṣkoṭi, of the last two options (iii) and (iv) coinciding? The Sierpiński topos, as it is known, is a case in point.

The objects of the Sierpiński topos are shortened versions of the configurations shown above: only two sets, B and C , rather than three, and only one function g. The truth value object is the right-hand fragment of the truth value object shown above, and has only three global elements, namely the three one-step paths emanating from the various elements of **3**, which have reason to be thought of as True, NeitherWhollyTrueNorNotWhollyTrue, and NotTrue (taken from top to bottom), respectively, though the middle value may equally well be thought of as TrueInTheLongRunEvenIfNotTrueAtTheOutset. This middle truth value, in other words, is at once BothTrueAndNotTrue and NeitherTrueNorNotTrue, and is the only alternative to the extreme values True and NotTrue.

3 Afterword

As a final topic, perhaps not worthy of even this passing mention, let us take up one objection on the part of some commentators to the tetralemma paradigm, namely, that there should by rights be yet a fifth alternative, something like NoneOfTheAbove, to the classical four. The Buddha himself, after all, is reported in one instance to have rejected, each in its turn, all four alternatives of one particular quadrilemma, indicating that the truth lay somehow elsewhere.

There are indeed topoi, readily described, whose global truth values

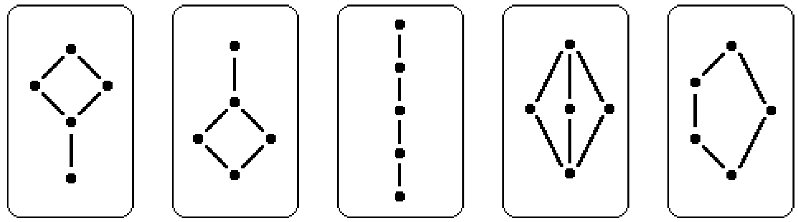


Figure 4:

easily realize the ideal of being five in number. For that matter, that ideal can be realized in three wholly different ways. In all cases, however, the lattice of global truth values must, for purely topos-theoretic reasons (that is, by virtue of what has been called generalized abstract nonsense), be what is known, to those in the lattice trade, as *distributive*. That requirement rules out the last two lattices depicted in Figure 4 above. The remaining five-element lattices number three: they too appear in Figure 4, as the first three on the left: they are all distributive, but none is Boolean.

And just which of their intermediate members (between True at the top and NotTrue at the bottom) should be interpreted as BothTrueAndNotTrue, as NeitherTrueNorNotTrue, or as NoneOfTheAbove, I leave as my parting conundrum to you.

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Brahmagupta's Bhāvanā: Some Reflections

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Abstract

We shall present Brahmagupta's treatment of the indeterminate equation $Dx^2 + 1 = y^2$ highlighting some ideas of modern algebra that are implicit in this ancient work of 628 CE and discuss the consequent pedagogic potential of Brahmagupta's results.

1 The Bhāvanā — An Introduction

Mathematics in India attained one of its highest peaks during the 7th century CE with the arrival of the versatile astronomer-mathematician Brahmagupta (born 598 CE). His major work *Brāhma Sphuṭa Siddhānta* ([B]), comprising over 1000 verses in 24 chapters, was composed in 628 CE. Two of these chapters (12 and 18), dealing exclusively with mathematics, were translated into English by Colebrooke in 1817 ([C]).

Brāhma Sphuṭa Siddhānta is probably the first ancient Indian text having a separate chapter (18) on algebra. A substantial portion of this chapter is devoted to solutions of indeterminate equations of the first and second degree. This includes partial solutions of the celebrated Pell's equation. Brahmagupta is the earliest known mathematician to have systematically investigated *integer* solutions of $Dx^2 + 1 = y^2$. In the process he discovered significant results on the more general equation $Dx^2 + m = y^2$ called *varga-prakṛti* (square-natured) in ancient India. In the Preface of his treatise on history of number theory, L.E. Dickson made a special mention of this work ([Di], p. xi):

> *It is a remarkable fact that the Hindu Brahmegupta in the seventh century gave a tentative method of solving $ax^2 + c = y^2$ in integers, which is a far more difficult problem than its solution in rational numbers.*

All the results of Brahmagupta on this topic are clever applications of a certain law of composition called "*bhāvanā* ". This principle can be formulated, in modern language and notations, as follows:

Theorem 1 (Brahmagupta's Bhāvanā)
The solution space of the equation $Dx^2 + m = y^2$ admits the binary operations

$$(x_1, y_1, m_1) \odot (x_2, y_2, m_2) = (x_1y_2 \pm x_2y_1, Dx_1x_2 \pm y_1y_2, m_1m_2).$$

In other words, if (x_1, y_1, m_1) and (x_2, y_2, m_2) are solutions of $Dx^2 + m = y^2$, then so are $(x_1y_2 + x_2y_1, Dx_1x_2 + y_1y_2, m_1m_2)$ and $(x_1y_2 - x_2y_1, Dx_1x_2 - y_1y_2, m_1m_2)$.

The consequent identities

$$({y_1}^2 - D{x_1}^2)({y_2}^2 - D{x_2}^2) = (Dx_1x_2 \pm y_1y_2)^2 - D(x_1y_2 \pm x_2y_1)^2$$

are now called Brahmagupta's identities. The result was rediscovered by Euler during the middle of the 18th century. Euler highlighted the result in his writings as "*theorema eximium*" (a theorem of capital importance), "*theorema elegantissimum*" (a most elegant theorem), etc.[1]

For brevity, we shall adopt a notation suggested by Weil ([W2], p 21). For a given positive integer D, $(p, q; m)$ will denote a triple of numbers satisfying $Dp^2 + m = q^2$. Thus Theorem 1 states:

$$(p,\ q;\ m) \odot (r,\ s;\ n) = (ps \pm qr,\ Dpr \pm qs;\ mn).$$

Ancient Indian algebraists had realised the importance of the two laws and used the special technical term *bhāvanā* (composition) — the formula obtained by taking the positive sign was called the *samāsabhāvanā*[2]

[1]([W2], p 284–285; [Se], p 168).

[2]From the Vedic era, addition has been called *samāsa* ("putting together") and the sum obtained *samasta* ("whole", "total", etc) — see [D3], p 226. Note that the prefix *sam* (together) is used like the Latin *con* expressing "conjunction", "union", etc.

or *yoga-bhāvanā* (additive composition) and the one obtained by taking the negative sign was called the *viśeṣa-bhāvanā* or *antara-bhāvanā* (subtractive composition). In the special case of equal roots and interpolators, the rule was called *tulya-bhāvanā* (composition of equals); the general case was called *atulya-bhāvanā* (composition of unequals).[3]

The sheer beauty apart, Theorem 1 has a technical power, a glimpse of which can be felt from the way it can be used to solve a difficult indeterminate equation like $92x^2 + 1 = y^2$ in a few steps. This example is mentioned in ([C], p 364) immediately after the verses describing Theorem 1.

Example 1 (Brahmagupta) *Solve the equation* $92x^2 + 1 = y^2$, *in integers.*

Solution. One readily observes that $92 \times 1^2 + 8 = 10^2$. Composing the triple $(1, 10; 8)$ with itself (by *samāsa-bhāvanā*), and dividing the resulting triple $(20, 192; 8^2)$ by 8^2, one obtains the triple $(\frac{5}{2}, 24; 1)$ which, when composed with itself, gives the integer triple $(120, 1151; 1)$. Thus $(120, 1151)$ is a solution of $92x^2 + 1 = y^2$. ∎

It is interesting to note that after stating this problem, Brahmagupta had used the phrase *kurvannāvatsarād gaṇakaḥ* — "One who can solve it within a year (is truly a) mathematician."

The Pedagogic Issue

What strikes a modern algebraist reviewing Brahmagupta's ingenious treatment of the equation $Dx^2 + m = y^2$ is an incredible sophistication in the very attitude towards an algebraic problem. Fundamental concepts and principles of modern algebra are implicit in the *bhāvanā* and its applications. But though the *ideas* are modern, the presentation does not involve the elaborate *language* of modern abstract algebra. This aspect of Brahmagupta's work makes it particularly relevant for the algebra training of "would-be creative mathematicians"[4] among students (at higher-secondary or first-year college level) who have already demonstrated their skill in high-school algebra.

[3] See ([Sh] p 5–8; [C] p 170–172; [DS] p 146–148).
[4] Modification of a phrase from Weil ([1], p 228).

It is generally agreed that for a fuller realisation of one's creative potential, "One should learn from the (works of the) Masters". But does a higher-secondary student have an immediate, realistic and rapid access to any of the Masters embodying, even partially, the spirit of modern algebra? We suggest that Brahmagupta's results and applications can be utilised to promote the mathematical maturity and creativity of a bright student who has completed high-school algebra and is on the verge of making the sharp (and rather abrupt) transition from the "classical algebra" of school mathematics to the "modern algebra" or "abstract algebra" of college mathematics.

Arrangement

Some readers would be well-informed about ancient Indian results on indeterminate equations, some might not be. Again, while the professional algebraist knows the subtleties involved in the *bhāvanā*, the richness and depth of the result might not be apparent to all historians. Keeping in mind the possible diverse background of the readers, the paper has been arranged as follows.

In Section 2 we highlight certain features of modern algebra implicit in the *bhāvanā* and its applications. Section 3 proposes the central theme of the paper — the pedagogic potential of aspects mentioned in Section 2 in the context of "abstract algebra".

The applications of Theorem 1 are revisited in Sections 4 and 5 chiefly from the pedagogic perspective: Section 4 describes the immediate applications of the *bhāvanā*, mostly by Brahmagupta himself, while Section 5 traces its influence on the *cakravāla*. Section 6 touches another thought-provoking topic for the students: the possible genesis of the *bhāvanā*.

Section 7 discusses the possible motivation for investigation of the equation $Dx^2+1=y^2$. Section 8 contains a few miscellaneous historical remarks on the *bhāvanā*.

To convey to the readers some flavour of the original "composition" (pun intended!), Appendix presents the transliteration of Brahmagupta's verses describing Theorem 1. For readers uncomfortable with Sanskrit, these verses are followed by a translation. As Brahmagupta's language is somewhat cryptic, a more lucid version of Theorem 1, due to Bhāskara II, has also been included in Appendix.

2 Bhāvanā and Modern Algebra

André Weil asserted ([W1], p 231–232):

> *An understanding in depth of the mathematics of any given period is hardly ever to be achieved without knowledge extending far beyond its ostensible subject-matter. More often than not, what makes it interesting is precisely the early occurrence of concepts and methods destined to emerge only later into the conscious mind of mathematicians; the historian's task is to disengage them and trace their influence or lack of influence on subsequent developments.*

The entire section of Brāhma Sphuṭa Siddhānta on the *Varga-Prakṛti* demonstrates a phenomenal wizardry in classical algebraic manipulation at an early stage in the history of symbolic algebra. But more exciting is the implicit occurrence of some of the simple but powerful ideas which characterise modern algebra. We mention below three related concepts that are embedded in the *bhāvanā*: binary composition, the multiplicativity of the norm function, the composition and property of a binary quadratic form. If we leave aside the set-theoretic language, the principle of binary composition is quite explicit — in fact, in ancient Indian mathematics, *bhāvanā* means "composition"! We shall also analyse Brahmagupta's methods and highlight in his approach the dynamic anticipation of a certain mathematics culture that eventually evolved much later, albeit on a more broad and firm foundation.

Binary Composition

Theorem 1 defines an intricate binary operation (*samāsa-bhāvanā*)

$$(x_1, y_1, m_1) \odot (x_2, y_2, m_2) = (x_1y_2 + x_2y_1, Dx_1x_2 + y_1y_2, m_1m_2)$$

on $S = \{(x, y, m) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \mid Dx^2 + m = y^2\}$ where $\mathbf{Z}$ denotes the set of integers. This sophisticated idea of constructing a binary composition on an abstractly defined unknown set is the quintessence of modern "abstract algebra". The idea occurs in Brahmagupta's work in a fluid amorphous form — it had not been crystallised in a precise set-theoretic framework.

Brahmagupta did not present the collection S as a single entity. But he had envisaged the key ingredient of a modern abstract structure: *binary composition.* If we exclude the four elementary arithmetic operations on usual numbers, the *bhāvanā* is perhaps the first conscious construction of a binary composition. Further, the binary operation is quite a complicated one: it involves *two* integral *triples* of *unknown roots.* Recall that even basic symbolic computations with unknown *roots* — treating them as if they were known quantities — is a fairly modern approach that emerged during the investigations on the general polynomial in *one* unknown.

Norm Function

The *bhāvanā* laws $\odot$ have an elegant interpretation in terms of the norm function — a very important concept in modern mathematics. Let $A = \mathbf{Z}[\sqrt{D}]$ $(= \{b + a\sqrt{D} | a, b \in \mathbf{Z}\})$. The norm function on A is the map $N : A \to \mathbf{Z}$ defined by

$$N(y + x\sqrt{D}) = (y + x\sqrt{D})(y - x\sqrt{D}) = y^2 - x^2D.$$

Brahmagupta's identity can be reformulated as the statement:

The norm function N is multiplicative, i.e.,
$N(\alpha\beta) = N(\alpha)N(\beta)\ \forall \alpha, \beta \in A.$

Consider the bijective maps $f, g : \mathbf{Z} \times \mathbf{Z} \to A$ defined by $f(x, y) = y + x\sqrt{D}$ and $g(x, y) = y - x\sqrt{D}$. Then the set $S = \{(x, y, m) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \mid Dx^2 + m = y^2\}$ is simply the graph of each of the functions

$N \circ f$ and $N \circ g$. Let $p : \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ denote the map defined by $p(x, y, z) = (x, y)$. Then $p|_S$ is a bijection from S to $\mathbf{Z} \times \mathbf{Z}$ and hence $\phi = f \circ p|_S$ and $\psi = g \circ p|_S$ are bijections from S to A. The multiplicative structure *bhāvanā* on S is essentially the multiplication in the ring A: the *samāsa-bhāvanā* on S is obtained by transferring the ring multiplication on A via ϕ and the *antara-bhāvanā* is obtained through the conjugate ψ.

It is possible that Brahmagupta discovered the identity through an algebraic manipulation which was, in essence, the verification of the structure preserving property of the norm function — the natural multiplication in A providing the precise formula (see the last paragraph of Section 6).

Binary Quadratic Form

$y^2 - Dx^2$ is a binary quadratic form with discriminant D. In the language of quadratic forms, Brahmagupta's identity says that two such forms (say $y^2 - Dx^2, v^2 - Du^2$) can be *composed* to yield another such form with discriminant D in a new pair of variables $(xv \pm yu, yv \pm Dxu)$. Again recall that the term *bhāvanā* means "composition"! The theory of composition of quadratic forms, an important and rich topic initiated by Gauss and Dirichlet, is still an active area of research.

Here one may also recall that Fermat's researches in number theory led him to a deep study of the binary quadratic form $y^2 - 2x^2$ which must have resulted in his realisation of the far-reaching importance of the equation $y^2 - Dx^2 = 1$. As is well-known, Fermat not only investigated Pell's equation but also inspired others to take interest in the problem.

Before mentioning another aspect of Theorem 1 in the context of quadratic forms, we recall a few definitions. Two quadratic forms $f(x, y)$ and $g(x, y)$ over an integral domain K are defined to be *equivalent* if there exists a homogeneous linear change of variables which takes the form f to the form g; more precisely, if there exists an invertible matrix A with entries in K such that $g(\mathbf{x}) = f(A\mathbf{x})$ where $\mathbf{x}$ denotes the vector $(x, y)^T$. Thus f and g are equivalent if if there exist a, b, c, d in K such that $g(x, y) = f(ax + by, cx + dy)$ and $ad - bc$ is a unit in K.

An element c in K is said to be *represented* by a binary form $f(x, y)$ if the value c is attained by f, i.e., if there exist $a, b \in K$ such that

$c = f(a, b)$. A quadratic form f is said to be *strongly multiplicative* if f is equivalent to cf for every unit c represented by f.

Theorem 1 can be viewed as the result:

> *The binary form $y^2 - Dx^2$ is strongly multiplicative (over rationals).*

For, if f denotes the form $y^2 - Dx^2$, c a *non-zero* number represented by f, say $c = b^2 - Da^2$, and $g = cf$, then $g(x, y) = (b^2 - Da^2)(y^2 - Dx^2)$. Now Brahmagupta's identity prescribes the substitution that has to be made to obtain $g(x, y) = f(u, v)$: namely, the homogeneous linear transformation given by $u = bx + ay, v = Dax + by$ which is invertible (since the determinant of Brahmagupta's transformation is $c \neq 0$). This version of Theorem 1 was generalised in 1965 by A. Pfister using, what are now called, "Pfister forms". Pfister's discovery opened up new directions in the theory of quadratic forms.

In his recent research monograph ([O]) in this area, Manuel Ojanguren begins Chapter 5 by quoting Brahmagupta's original Sanskrit verses describing Theorem 1; the chapter itself is titled **Also sprach Brahmagupta**[5]. The first result in the chapter (Lemma 5.1) states:

> If K is a field of characteristic different from 2, and c is a non-zero element of K represented by the quadratic space $< 1, a >$, then $< 1, a >$ is isometric to $< c >< 1, a >$.

Readers familiar with the relevant terminology can see that Lemma 5.1 is a version of the fact that $y^2 - Dx^2$ is strongly multiplicative. M. Ojanguren referred to Lemma 5.1 as "Brahmagupta's lemma" ([O], p 55).[6] Theorem 5.2 in ([O], p 55) states the generalisation of Lemma 5.1 by Pfister: *Pfister forms are multiplicative.*

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[5]In English: "Thus Spake Brahmagupta"!

[6]Just before the statement of Lemma 5.1, Ojanguren writes ([O], p 54): "In [5, Ch.18] Brahmagupta proves, in the slightly different formulation quoted above, the following lemma.". With a quiet humour characteristic of Ojanguren, Item 5 is listed in "References" as: 5. Brahmagupta: Brāhmasphuṭasiddhānta, Bhīnmāla, 628 (preprint).

Quest for General Algebraic Principles

One profound message of modern algebraic research is that an isolated problem can often be handled more effectively by viewing it as part of a larger set-up. A deep understanding of the general picture not only generates additional techniques for the special situation, it also leads to new creations and discoveries whose overall impact could be of much greater value than a mere solution to the original problem. This realisation has heralded a new attitude in algebra: the search for general principles. Lagrange (1770 CE) pioneered the trend. The culture of generalisation has been diligently pursued in 20th century algebra.

Now let us consider certain aspects of Brahmagupta's ancient work in this light. The example $D = 92$, mentioned in Section 1, serves as an illustrative model.

(i) While trying to solve a specific hard problem $Dx^2 + 1 = y^2$ (in two variables), Brahmagupta undertook a bold and farsighted exploration of the *general* picture: the solution space of $Dx^2 + m = y^2$ (in three variables). In the process he discovered and extracted an important general and abstract principle (Theorem 1), and made a clear enunciation of this principle for posterity. It is amazing that in an attempt to solve an indeterminate equation in two variables, a seventh-century mathematician thought of constructing, what amounts to, an intricate abstract structure on the solution space of an equation in three variables. This is an original attitude in mathematics the like of which was not to be seen for the next 1000 years.

(ii) The power of the general principle can be seen from the deceptive ease with which it immediately provides the solution of a difficult equation like $92x^2 + 1 = y^2$ (Recall Example 1).

(iii) Apart from providing partial solutions to the original problem (see Section 4), the *samāsa-bhāvanā* principle contained the key to the the subsequent complete solution (see Section 5).

(iv) In retrospect, one realises that the *samāsa-bhāvanā* had also opened up new possibilities discussed in Section 2. However, it was far too ahead of the times to have immediate fruition beyond the solution of Pell's equation. Its true potential began to be harnessed only after its rediscovery in the 18th century. This general principle turned out to be "a theorem of capital importance" and even a starting point of an area in algebra.

A Caveat

Sometimes the brilliance of an algebraic research lies in its opening up of new and unexpected horizons with immense possibilities through surprisingly simple innovations. Ironically, the very simplicity of the work tends to hinder or distract later generations from a deep perception of its true worth. A casual observer could very well fail to fully appreciate the richness and magnificence of the ideas encapsuled in the construction of a mere 2-step solution of $92x^2 + 1 = y^2$. Further, once the mathematical community gets accustomed to an original idea like the *bhāvanā*, it becomes all the more difficult to fathom the greatness of the discovery. Historians need to be aware of this intrinsic risk of missing the real depth and significance of Brahmagupta's work on the *Varga-Prakṛti*.

3 Pedagogic Potential of Bhāvanā

We first clarify the aspect of mathematics education which we shall be addressing. Our target is the (potentially) research-oriented higher-secondary or first-year college student who is about to learn modern algebra. We focus on two related aspects: (1) promotion of a culture of creativity in the introduction to abstract structures and (2) promotion of "the art of discovery".

Training in Abstract Algebra

During his first encounter with "abstract algebra", the student of mathematics is suddenly confronted with a formidable edifice of axiomatic structures. He learns systematically, but more or less passively, a large number of definitions and basic properties. For the student, this "algebra" has not much apparent connection with the manipulative high-school algebra that he hitherto enjoyed. In fact, with passage of time, he even tends to lose his classical manoeuvring skill as his training usually focusses, almost exclusively, on understanding of abstract structures. During this transitional period, the student has hardly any chance to mentally participate in the process of discovery of the basics of modern algebra — he has to patiently learn a radically new approach to mathematics suspending his creative impulse. The situation is all the more grave for the large number of students who do not have scope for interaction with creative algebraists.

The current group-ring-field approach of abstract algebra is undoubtedly neat, elegant and has enormously simplified mathematics. But simplicity tends to hide mathematical subtleties. Abrupt introduction to abstract structures, completely divorced from their original contexts, could stifle the natural growth of thought-process and promote a sort of mechanical pursuit of forms missing the substance. The achievement of extreme elegance and simplicity, therefore, has a potential risk for new-generation learners. They tend to approach algebra with a mechanised mindset for too long.[7]

Is it possible to present before a high-school student a non-trivial but accessible algebraic work of a mathematical genius which can creatively orient him to the principles of modern algebra through the language of high-school algebra?

We affirm that, before the student gets lost in the elaborate maze of groups, rings, fields, vector spaces, etc, Brahmagupta's results can be used to informally introduce him to the essence of abstract algebraic ideas. An imaginative and effective use of Theorem 1 and Example 1 can convey to the fresh student the power of "binary composition" much more deeply and rapidly than any of the standard pedagogic approaches. It could be inspiring for him to realise that the simplicity of the solution to Example 1 has its secret in the monoid structure of $(S, \odot)$ of Section 2 which in turn has its root in the natural ring structure of the set A in Section 2. He could be excited by the observation that, through *samāsa-bhāvanā*, the set $\{(x, y, m) \in \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} | Dx^2 + m = y^2\}$ forms a group or the later realisation that $\{(x, y) \in \mathbf{N} \times \mathbf{N} | Dx^2 + 1 = y^2\}$ forms

[7] As the inevitable perils are gradually emerging, there is now an increasing trend to supplement the organised and rapid presentation of abstract algebra in its generality with some discussion on the genesis of the subject. It is a welcome sign that attempts are being made in recent texts to impart some familiarity with the original approach of Galois supplementing the standard 20th century presentation. But considerable preparation is needed before one can get introduced to fragments of the thought-process of a Lagrange or a Galois.

a cyclic monoid. What is more, the unexpected invocation of abstract algebraic technique for a concrete problem by an ancient mathematician could infuse in him a dynamic and creative vigour in his formal study of abstract algebra. Thus, through this deep but easily accessible work of a Master, the perceptive student can get an early exposure to modern sophistication, see a natural application of abstract algebraic principles, develop a flair for exploring worthwhile generalisation, and truly imbibe the spirit of the abstract algebra culture without getting swept away by its formalism.

The Art of Discovery

In his plenary lecture at the ICM (1978), André Weil quoted the following statement of Leibniz on history of mathematics:

> *Its use is not just that History may give everyone his due and that others may look forward to similar praise, but also that the art of discovery be promoted and its methods known through illustrious examples.*[8]

Weil added ([W1], p 229):

> *Deviating only slightly from Leibniz, we may say that its first use for us is to put or to keep before our eyes "illustrious examples" of first-rate mathematical work.*

For our target students, the developments on Diophantine equations in ancient India — the *kuṭṭaka* of Āryabhaṭa and his successors, the *bhāvanā* of Brahmagupta and the *cakravāla* of Jayadeva-Bhāskara — could be utilised to promote the "art of discovery". Especially, Brahmagupta's *bhāvanā* would serve not only as an "illustrious example of first-rate mathematical work" but also as an early example of modern methods and techniques and of the harmonious blending of the classical and modern algebra styles. It would help students realise how the seeds of various abstract structures lie in earlier classical algebra.

Brahmagupta's *bhāvanā* is the first known instance of an involved abstract algebraic thinking. Its pedagogic value stems from the fact

[8]Both the original and the above translated version occur in ([W1], p 227).

that it is a self-contained gem. When the same ideas resurfaced in Europe after more than a thousand years, they came as part of big theories developed by a large number of mathematical giants. The raw student needs a long time to assimilate even the subsequent simplified versions of these theories. By contrast, the applications and influence of *bhāvanā* (discussed in next two sections), especially the *cakravāla*, are ready-made materials for promoting the art of discovery. The student need not have prior preparation and the volume of work he has to dwell on is well within a manageable limit.

Note that the solution of Pell's equation is taught quite neatly in several texts on classical algebra and elementary number theory. But those presentations offer a finished product to the students. Continued fraction is defined, its properties are systematically established and, in due course, the solution of Pell's equation is described. The approach has its usefulness but does not serve the purpose alluded to by Leibniz and Weil.

To look at the pedagogic aspect from another angle, a research scholar gets trained from two sources: text books, research papers. They serve complementary purposes. By and large, school and college students have access only to the former. Can one give them some material which fulfils one aspect of the role played by research papers at a later stage?

We affirm that robust expositions on the *kuṭṭaka*, *bhāvanā*, *cakravāla*, etc, in the style of modern expository research papers, can play that role. A preliminary attempt was made in that direction by the present author in ([Du1], [Du2]) for a journal on science education.[9] Each of these articles contained an introductory part meant to excite or inspire the student on the topic; but the major part of the articles tried to give some flavour of research culture.

Besides, to use a phrase of Weil ([W1], p 231) made in a slightly different context, the students could also gain by the encouragement to

> *recognize mathematical ideas in obscure or inchoate form, and to trace them under the many disguises which they are apt to assume before coming out in full daylight.*

[9] The pedagogic usefulness of *bhāvanā* was also perceived by M. Ojanguren. After receiving a draft of ([Du2]), Ojanguren mentioned in an e-mail to the author that he thought of using the results in an elementary course for school teachers.

4 Immediate Applications of Bhāvanā by Brahmagupta

Brahmagupta's presentation of the results on *Varga-Prakṛti* resembles a typical modern arrangement where one first develops the fundamental theory or principle that emerges out of the investigations into a problem, and then records the solution to the original problem(s) as one application of the basic theory. Thus, after announcing *Atha Varga Prakṛtiḥ*, Brahmagupta immediately states Theorem 1 — the cornerstone of the section on *Varga-Prakṛti* — in Verses 64–65 of ([1], Ch. 18). This is followed by a few general results on the *varga-prakṛti* which are useful offshoots of Theorem 1: construction of infinitely many rational solutions of $Dx^2 + 1 = y^2$ (Verse 65), generation of infinitely many integer solutions of $Dx^2 + m = y^2$ from a given one (Verse 66), derivation of solutions of $Dx^2 + 1 = y^2$ from solutions of $Dx^2 \pm 4 = y^2$ (Verses 67–68), etc. The results are then illustrated by several concrete numerical examples. The example $D = 92$ (Example 1) is mentioned in Verses 71–72.

Application 1: Rational Solutions of $Dx^2 + 1 = y^2$.

The verses 64–65 in ([1], Ch. 18) contain, apart from Theorem 1, the following two corollaries (see Appendix).

Corollary 1 (Brahmagupta)
If $Dp^2 + m = q^2$, then $D(2pq)^2 + m^2 = (Dp^2 + q^2)^2$.

Corollary 2 (Brahmagupta)
If (p, q) is a root of $Dx^2 \pm c = y^2$, then $(\frac{2pq}{c}, \frac{Dp^2+q^2}{c})$ is a root of $Dx^2 + 1 = y^2$.

In fact, immediately after stating their respective verses describing Theorem 1, Brahmagupta, Jayadeva ([Sh], p 8) and Bhāskara II ([Ba], p 23) describe non-trivial rational solutions of $Dx^2 + 1 = y^2$.

This is actually an easy application — the full strength of Theorem 1 is not really needed. One can always choose a positive integer p (for instance, $p = 1$) and a positive integer q such that $q^2 > Dp^2$ and put $m = q^2 - Dp^2$ to get $Dp^2 + m = q^2$ and then apply Corollary 2. The latter needs Corollary 1 which could be discovered directly. In fact, finding a non-trivial rational solution of the Pell's equation could itself be set as a reasonable challenge to the student inciting him to directly discover Corollaries 1 and 2.

Brahmagupta simply made a terse statement amounting to Corollary 2 (see Appendix) but did not bother to give the trivial explanation as to how to obtain some integer-triple $(p, q; m)$. As in modern times, one cannot always expect the Masters, who have to offer volumes of original materials, to spell out all obvious details. (See Remark 4 in Appendix.)

But as the history of algebra from the 1870s will testify, clarity of exposition can do wonders for the rapid progress of a subject. In ancient Indian algebra, the terseness of Brahmagupta was balanced by the clearer accounts of later expositors. For instance, Śrīpati (1039) described the method for obtaining rational solutions as follows:

"Unity is the lesser root. Its two squares [set at two places] are [each] multiplied by the *prakṛti* [and the product is] decreased [by the *prakṛti* and] increased by a [suitable] interpolator whose square-root will be the greater root. From these two, two roots are obtained by *bhāvanā*. There will be an infinite [set of two roots]."[10]

In other words, applying Brahmagupta's *bhāvanā* on two copies of the identity

$$D.1^2 + (q^2 - D) = q^2,$$

one gets a rational solution

$$x = \frac{2q}{q^2 \sim D}, \qquad y = \frac{q^2 + D}{q^2 \sim D}$$

of $Dx^2 + 1 = y^2$. Infinitely many roots can be obtained by varying q (also by repeated use of *bhāvanā*).

The above solution was also described by Bhāskara II (1150), Nārāyaṇa (1350) and later writers like Jñānarāja (1503), Kṛṣṇa (1600) and Kamalākara (1658). To quote Bhāskara II:

"Divide twice an optional number by the difference between the square of that optional number and the *prakṛti*. This [quotient] will

[10] Verse 33 of Chapter 14 (on algebra) of the astronomy treatise Siddhāntaśekhara; quoted (in translation) in ([Si], p 40; [DS], p 153, 150).

be the lesser root [of a *varga prakṛti*] when unity is the additive. From that [follows] the greater root."[11]

Now the application of *bhāvanā* on the identity $D.1^2 + q^2 - D = q^2$ to obtain rational solutions might appear to be a relatively minor feat in the history of Indian algebra. But the clear formulation of this identity, the display of the action of *bhāvanā* on it, and explicit mention of division by $q^2 - D$ might have facilitated the invention of the *cakravāla* (see next section). We can also expect that, during actual computation, there would be a tendency to choose q so as to avoid division by large numbers. This leads to a q for which $|q^2 - D|$ is the minimum — another component of the complex *cakravāla*. In this connection, one is reminded of Weil's caveat ([W1], p 235):

> *It is also necessary not to yield to the temptation (a natural one to the mathematician) of concentrating upon the greatest among past mathematicians and neglecting work of only subsidiary value. ... historically it can be fatal, since genius seldom thrives in the absence of a suitable environment, and some familiarity with the latter is an essential prerequisite for a proper understanding and appreciation of the former. Even the textbooks in use at every stage of mathematical development should be carefully examined in order to find out, whenever possible, what was and what was not common knowledge at a given time.*

However not enough of historical materials have been found to ascertain the actual sequences in emergence and flow of ideas. (See Section 8.)

Application 2: Generation of Infinitely Many Roots

Verse 66 in ([B], Ch 18) can be stated as:

Theorem 2 (Brahmagupta) *If the equation $Dx^2 + m = y^2$ has one positive integral solution, it has infinitely many.*

The general equation $Dx^2 + m = y^2$ need not have any integral solution (students could be encouraged to find simple examples like

[11] Verse 73 of Bījagaṇita ([Ba], p 23); quoted (in translation) in ([DS], p 154).

$D = 3, m = 2$). However, when it does have an integral solution, Brahmagupta's verse 66 prescribes the *samāsabhāvanā* to use that solution and a non-trivial integral solution of $Dx^2 + 1 = y^2$ to generate infinitely many integral solutions of $Dx^2 + m = y^2$. In particular, from one positive integral solution of $Dx^2 + 1 = y^2$, one gets infinitely many. The verses 65–66 tacitly describe infinitely many rational solutions of $Dx^2 + 1 = y^2$. This consequence is mentioned explicitly in Śrīpati (quoted earlier) and Bhāskara II.

It would be interesting for students to observe that the method for generating arbitrarily large solutions of $Dx^2 + 1 = y^2$ was useful for determination of rational approximation to $\sqrt{D}$. If $Da^2 + 1 = b^2$, then

$$\frac{b}{a} - \sqrt{D} = \frac{b^2 - Da^2}{a(b + a\sqrt{D})} = \frac{1}{a(b + a\sqrt{D})}.$$

Thus, for a sufficiently large solution (a, b), $\frac{b}{a}$ will be a good approximation for $\sqrt{D}$. This application was explicitly stated by the algebraist Nārāyaṇa around 1350 ([D2], p 187–188). By that time, the *cakravāla* method for determination of one (in fact, the minimum) positive integral solution had already been invented —- Nārāyaṇa himself was an expositor of both the *bhāvanā* and the *cakravāla*. Nārāyaṇa illustrated the method for successive approximations of surds by two numerical examples: $\sqrt{10}$ and $\sqrt{\frac{1}{5}}$. For $\sqrt{10}$, he mentioned the rational approximations $\frac{19}{6}$, $\frac{721}{228}$ and $\frac{27379}{8658}$. For $D = 10$, since 9 is the nearest square, one has the initial triple $(1, 3; -1)$. Now,

$$(1, 3; -1) \odot (1, 3; -1) = (6, 19; 1),$$

$$(6, 19; 1) \odot (6, 19; 1) = (721, 228; 1),$$

$$(6, 19; 1) \odot (721, 228; 1) = (27379, 8658; 1)$$

and hence the three successive fractions of Nārāyaṇa. The case of $\sqrt{\frac{1}{5}}$ is similar. This method of getting successively closer approximations was restated by Euler in 1732 ([D2], p 188).

Application 3: Integer Solution of $Dx^2 + 1 = y^2$ (Partial)

Verses 67–68 in ([B], Ch 18) give the following consequences of the *samāsabhāvanā*.

Theorem 3 (Brahmagupta)

(i) If $Dp^2 + 4 = q^2$, *then* $(\frac{1}{2}p(q^2 - 1), \frac{1}{2}q(q^2 - 3))$ *is a solution of* $Dx^2 + 1 = y^2$.

(ii) If $Dp^2 - 4 = q^2$, *and* $r = \frac{1}{2}(q^2+3)(q^2+1)$, *then* $(pqr, (q^2+2)(r-1))$ *is a solution of* $Dx^2 + 1 = y^2$.

Since $(\pm 1)^2 = 1$ and $(\pm 2)^2 = 4$, it follows that from any positive integer solution of $Dx^2 + m = y^2$, where $m \in \{-1, \pm 2, \pm 4\}$, one can derive a positive integer solution of $Dx^2 + 1 = y^2$ by repeated use of the *samāsabhāvanā*. This consequence was explicitly recorded by Śrīpati ([Si], p 40 ;[DS], p 157). Brahmagupta applied it to numerical examples. We quote two of them ([B], Ch 18, Verses 71–72, 75).

Example 2 (Brahmagupta) *Solve, in integers,* $13x^2 + 1 = y^2$.

Solution. $13 \times 1^2 - 4 = 3^2$. Now Theorem 3 yields the solution $(180, 649)$. ■

Brahmagupta mentioned the equations $13x^2 + 300 = y^2$ and $13x^2 - 27 = y^2$. By inspection, one has solutions $(10, 40)$ and $(6, 21)$ respectively. Composing with $(180, 649)$ one gets larger solutions for the two equations.

Example 3 (Brahmagupta) *Solve, in integers,* $83x^2 + 1 = y^2$.

Solution. $83 \times 1^2 - 2 = 9^2$. $(1, 9; -2) \odot (1, 9; -2) = (18, 164; 4)$. As the quantities are all even, dividing by 2^2, one gets the triple $(9, 82; 1)$, i.e., the solution $(9, 82)$. ■

Brahmagupta stated the formulae for $D = \pm 4$ in a generality required for the harder case: p odd. Students could be encouraged to deduce Theorem 3 from Theorem 1 and to find simpler formulae when p is even as also the formulae for $D = -1$ and $D = \pm 2$. They should be able to discover results like $(p, q; -1) \Rightarrow (2pq, 2q^2 + 1; 1)$, $(p, q; \pm 2) \Rightarrow (pq, q^2 \mp 1; 1)$; and, for p even, $(p, q; 4) \Rightarrow (p/2, q/2; 1)$, $(p, q; -4) \Rightarrow (pq/2, q^2/2 + 1; 1)$. Brahmagupta used these simple formulae in his examples though he did not state them explicitly.

Theorem 3 already enabled Brahmagupta to solve various difficult cases like $D = 83$ or $D = 92$. In the next section, we shall see, with the example of the famous case $D = 61$, how it accelerated the subsequent *cakravāla* algorithm.

5 Role of Bhāvanā in Cakravāla

Brahmagupta attained partial success in his attempt to solve $Dx^2+1 = y^2$. His results can directly be applied only to those specific values of D for which one gets some integer triple $(p, q; m)$, where $m \in \{\pm1, \pm2, \pm4\}$, through inspection or clever manipulations. Even when such a triple becomes available for a special D, the methods lead to *some* positive integral solution of $Dx^2+1 = y^2$ — but not necessarily the *minimum* and hence although the *samāsabhāvanā* would fetch *infinitely* many integral solutions, it might not fetch *all* integral solutions. For instance, for $D = 3$, an application of Brahmagupta's formula (for $m = 4$) on the simple identity $3 \times 2^2 + 4 = 4^2$ yields the triple $(15, 26; 1)$; but the minimum triple (for $D = 3$) is $(1, 2; 1)$.

But Brahmagupta's partial solution, apart from being a remarkable landmark by itself, was also a significant step towards the celebrated *cakravāla* algorithm — a perfect method (free from trial-and-error) for obtaining, for *any* D, all positive integral solutions of $Dx^2 + 1 = y^2$. Actually the set of positive integral solutions of $Dx^2 + 1 = y^2$ forms a cyclic monoid under the operation *samāsabhāvanā* and the *cakravāla* method fetches the minimum solution, i.e., the generator of the above monoid. Thus, after the application of *cakravāla*, the *samāsabhāvanā* generates *all* integral solutions from the minimum one.

Historians have paid glowing tributes to the *cakravāla*. H. Hankel exclaimed[12]:

> *It is beyond all praise; it is certainly the finest thing achieved in the theory of numbers before Lagrange.*

More recently, Selenius[13] opined ([Se], p 180):

[12] Zur Geschichte der Mathematik in Alterthum und Mittelalter (Leipzig, 1874), p 202; quoted in ([Bg], p 28; [Se], p 170).

[13] C.O. Selenius gave a formal justification of the *cakravāla* in terms of his gener-

> *The old Indian cakravāla method for solving the mathematically fundamental indeterminate varga-prakṛti equation [$Dx^2 + 1 = y^2$] was a very natural, effective and labour-saving method with deep-seated mathematical properties. ... More than ever are the words of Hankel valid, that the cakravāla method was the absolute climax ("ohne Zweifel der Glanzpunkt") of old Indian science, and so of all Oriental mathematics.... no European performances in the whole field of algebra at a time much later than Bhāskara's, nay nearly up to our times, equalled the marvellous complexity and ingenuity of cakravāla.*

The earliest known author on the *cakravāla* is Ācārya Jayadeva.[14] His verses on *cakravāla* have been quoted in ([Sh]). Bhāskara II [15] (1150) too described the algorithm in ([Ba], Verse 75).[16]

The underlying idea of this brilliant algorithm can be put as follows: From a triple $(p_n, q_n; m_n)$ such that $|m_n|$ is "small", one has to construct a triple $(p_{n+1}, q_{n+1}; m_{n+1})$ with $|m_{n+1}|$ "small" eventually arriving at $(p, q; 1)$.

As we remarked in the preceding section, triples of the type $(1, y; y^2 - D)$ seem to have been "in the air" among Indian algebraists who expounded on Brahmagupta's work. The authors of *cakravāla* begin with an initial triple $(p_0, q_0; m_0)$ of this type: $p_0 = 1$, q_0 a number for which $|{q_0}^2 - D|$ is minimum, and $m_0 = {q_0}^2 - D$.

Now having arrived at an integer triple $(p_n, q_n; m_n)$, it would be natural to compose it with the (variable) triple $(1, y; y^2 - D)$ and explore the outcome. The *samāsabhāvanā* yields an integer triple $(p_n y + q_n, Dp_n +$

alised continued fraction expansion. He gave a talk on it in a Short Communications session of the "Algebra and Theory of Numbers" section of the 1962 ICM at Stockholm.

[14] Jayadeva's verses on the complete solution of the indeterminate equation $Dx^2 + 1 = y^2$ have been quoted in the text *Sundarī* of Udayadivākara composed in 1073 CE. This text itself was discovered only in 1954 by K.S. Shukla. Nothing is known so far about Jayadeva except for the obvious fact that he lived before 1073 CE. The algebraist Jayadeva is not to be confused with the Vaiṣṇava poet of the 12th century who composed Gīta-Govinda.

[15] Also known as Bhāskarācārya.

[16] Bhāskara's version has been quoted in the paper "Algorithms in Indian Mathematics" by M.S. Sriram in this volume.

$q_n y; m_n(y^2 - D))$. Dividing by ${m_n}^2$ (another natural idea given the earlier emphasis on methods for obtaining rational solutions) one obtains a rational triple $(p_{n+1}, q_{n+1}; m_{n+1}) = (\frac{p_n y + q_n}{m_n}, \frac{Dp_n + q_n y}{m_n}; \frac{y^2 - D}{m_n})$. Now the above triple would be an integer triple if y can be so chosen such that p_{n+1} (and hence q_{n+1}, m_{n+1}) becomes an integer. This amounts to finding integer solutions of the equation $m_n x - p_n y = q_n$. The linear indeterminate equation had been extensively discussed by Indian algebraists from the time of Āryabhaṭa (499 CE) and they knew the complete solution to the problem. Now among the infinite solutions, one would obviously choose the solution for which $|y^2 - D|$ would be minimum so that $|m_{n+1}|$ is minimised (as desired). And that is precisely the prescription of the *cakravāla*! André Weil's remark can hardly be bettered ([W2], p 22):

> *As is the case with many brilliant discoveries, this one [cakravāla] can be seen in retrospect as deriving quite naturally from the earlier work [samāsabhāvanā].*

Weil had written in a different context ([W1], p 231):

> *...in large part the art of discovery consists in getting a firm grasp on the vague ideas which are "in the air", some of them flying all around us, some (to quote Plato) floating around in our own minds.*

It would be instructive for the students to see that Brahmagupta's novel ideas contained the key to the discovery of the *cakravāla* and that the astonishing algorithm was possibly the result of a marvellous interplay between the two great preceding works — the *kuṭṭaka* and the *bhāvanā* — perhaps catalysed by the clarity brought about by expositions on lesser results like rational solutions of *varga-prakṛti*.

Selenius expressed doubt ([Se], 176–177) whether the original inventors of *cakravāla* actually thought in terms of the *bhāvanā*. In the absence of historical details, one can never be sure regarding the chain of thoughts involved in the discovery. But this reconstruction is, in any case, useful for our pedagogic purpose. It is historically plausible given that the verses on *cakravāla*, in both Jayadeva and Bhāskara, are preceded by verses on the *bhāvanā*; and that the *bhāvanā* was highly regarded by Indian algebraists including Jayadeva and Bhāskara — Jayadeva remarking that the *bhāvanā* pervades numerous algorithms. The alternative interpretation of Selenius, in the framework of continued fractions, too has its pedagogic merits in promoting the culture of creativity.

In retrospect, the original *cakravāla* can be simplified by avoiding the *kuṭṭaka*.[17] Denote by (x_n, y_n) the solution of $m_n x - p_n y = q_n$ for which $|y^2 - D|$ is minimal. Then $y_{n+1} + y_n \equiv 0 \pmod{m_n}$. Thus the sequence y_n can be inductively constructed as follows: having constructed y_n, y_{n+1} is defined to be an integer y satisfying the simple congruence $y \equiv -y_n \pmod{m_n}$ for which $|y^2 - D|$ is minimal.

Simplifications are observed *after* initial breakthroughs. For the purpose of promoting the art of discovery, it is desirable to highlight the initial thought-process which actually achieved the breakthrough and which tends to get camouflaged in later simplifications. There is a certain richness of thought in the cute approach through the blending of *kuṭṭaka* and *bhāvanā* which would be good for the students to imbibe. Of course, they should also see the subsequent simplification which too serves a lesser, but still important, pedagogic purpose.

The *bhāvanā* also aids the *cakravāla* in rapidly arriving at the minimum solution. Let us recall the famous example of Bhāskara ([Ba], Verse 76):

Example 4 (Bhāskara II) *Solve, in positive integers,* $61x^2 + 1 = y^2$.

Solution. As 64 is the perfect square nearest to 61, we have the initial triple $(1, 8; 3)$. Now one finds positive integer y for which $\frac{y+8}{3}$ is an integer and $|y^2 - 61|$ is minimised. Clearly $y = 7$. Now

$$\frac{7+8}{3} = 5; \frac{61 + 8 \times 7}{3} = 39; \frac{7^2 - 61}{3} = -4.$$

Thus we have the second triple $(5, 39; -4)$. Now rather than continuing the *cakravāla*, one can apply Brahmagupta's formula (Theorem 3) on the triple $(5, 39; -4)$ to immediately get the minimum positive solution $(226153980, 1766319049)$ of $61x^2 + 1 = y^2$. ■

[17] This observation is mentioned in the pioneering work of A.A. Krishnaswamy Ayyangar ([K], p 234).

As displayed in the two tables in Sriram's paper in this volume, the Brouncker-Euler algorithm needs 22 steps while the *cakravāla* algorithm (without the *bhāvanā*) needs 14 steps. Thus, although the "cyclic" character of *cakravāla* gets destroyed (making theoretical justification more complicated), the *bhāvanā* provides a very effective short-cut for the purpose of practical numerical computations.

In 1657, Fermat, in an effort to arouse the interest of contemporary mathematicians to number theory, issued a few challenge problems with special emphasis upon the problem of finding integer solutions to the equation $Dx^2+1=y^2$, unaware of the works of Brahmagupta-Jayadeva-Bhāskara. Regarding the challenge, André Weil remarked ([W2], p 81–82):

> *What would have been Fermat's astonishment if some missionary, just back from India, had told him that his problem had been successfully tackled there by native mathematicians almost six centuries earlier!*

It is interesting to note that Fermat specifically mentioned the case $D=61$ which was used by Bhāskara to illustrate the *cakravāla*.

6 Possible Genesis of the Bhāvanā

It would also be useful for a budding researcher to try to imagine, in retrospect, how a great mathematician could possibly have arrived at his discovery. Even if the actual thought-process had been completely different, meaningful speculations can serve an important purpose by providing fresh mathematical insights to students. We briefly summarise the discussion on the genesis of *bhāvanā* made in this light in ([Du2], p 20–21).

It is well-known that the Vedic fire-altar construction problems often involved the so-called Pythagorean triples — integer triples (x, y, z) satisfying $x^2+y^2=z^2$. The Śulba Sūtras mention several such triples. It is possible that the Śulba authors were aware of the formula $(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2$.[18] It is likely that, while handling Pythagorean triples,

[18] This identity can be obtained from the algebraic formulae $pq=(\frac{p+q}{2})^2-(\frac{p-q}{2})^2$

ancient Indians investigated the more general indeterminate equation $z = x^2 + y^2$ and discovered the algebraic formulae[19]

$$({x_1}^2 + {y_1}^2)({x_2}^2 + {y_2}^2) = (x_1x_2 \pm y_1y_2)^2 + (x_1y_2 \mp x_2y_1)^2. \qquad (*)$$

Now Brahmagupta explicitly gave the integer solution $(2mn, m^2 - n^2, m^2 + n^2)$ of the equation $x^2 + y^2 = z^2$ ([B], Chap 12, Verse 33; [C], p 306). It is thus possible that Brahmagupta had also examined $x^2 + y^2 = c$, arrived at the result $(*)$ and, then, while considering the more difficult $Dx^2 + m = y^2$, he could have been on the lookout for analogous formulae!

Brahmagupta's identity readily follows from $(*)$ by replacing x_1 by $x_1\sqrt{-D}$ and x_2 by $x_2\sqrt{-D}$. Weil remarks ([W2], p 14) that "this could not have been fully realized until the eighteenth century". Complex numbers appeared in formal mathematics only in the 16th century CE and its vigorous use can be seen only from the 18th century.

True, that validity of complex numbers had not been recognised in ancient Indian mathematics. But it is still possible that Brahmagupta might have heuristically used them as a secret trick — an inspired "rough work" — for guessing the magic identity which could then be verified by recognised algebraic manipulations.[20]

Brahmagupta's identity $(y^2 - Dx^2)(t^2 - Dz^2) = (yt + Dxz)^2 - D(yz + tx)^2$ may also be obtained by splitting the terms $y^2 - Dx^2$, $t^2 - Dz^2$, observing the identity $(y + x\sqrt{D})(t + z\sqrt{D}) = (yt + Dxz) + (yz + xt)\sqrt{D}$, and multiplying this identity by the conjugate identity $(y - x\sqrt{D})(t -$

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as well as $ka^2 = (\frac{k+1}{2})^2a^2 - (\frac{k-1}{2})^2a^2$ (putting $p = m^2, q = n^2$; $a = n^2, k = m^2/n^2$ and clearing denominators); and these two formulae are involved in the Śulba transformation of a rectangle into a square (of equal area) and Kātyāyana's ingenious rule for combining k squares into a single square. (See [D3], p 133–136.)

[19]This surmise finds some support from parallel developments in Greek mathematics. The Greeks too were fascinated by the Pythagorean triples — both Euclid (300 BCE) and Diophantus (c. 250 CE) gave the general solution to the Pythagorean equation — and there are indications that Diophantus was familiar with the identity (*). For, Diophantus mentioned that 65 is expressible as sum of two different squares in two different ways since 65 is the product of 13 and 5, each of which is a sum of two squares ([W2], p 10–11).

[20]To give an analogy, recall the non-commutative ring-theoretic result: "If $1 - ab$ is invertible, then so is $1 - ba$." and consider the effective trick of *guessing* the inverse of $1 - ba$ through power-series expansion of $(1 - ba)^{-1}$ even when the latter does not make formal mathematical sense.

$z\sqrt{D}) = (yt + Dxz) - (yz + xt)\sqrt{D}$. This was the approach of Euler ([W2], p 15). One wonders if Brahmagupta's thought-process too had taken a similar route! That would amount to an implicit handling of the norm function mentioned in Section 2.

7 Possible Motivation for Varga-Prakṛti

Pell's equation is the most famous Diophantine equation after $x^2 + y^2 = z^2$. Its importance in the study of binary quadratic forms and real quadratic fields is well-known. But what prompted Brahmagupta's vigorous pursuit of $Dx^2 + 1 = y^2$ in the seventh century?

Different explanations can be offered on the basis of the scarce historical materials available — each instructive in its own way.

(i) Astronomy

In *Laghu-Bhāskarīya* (Chap 8; Verse 18) of Bhāskara I, there is a problem from astronomy involving the simultaneous equations

$$8x + 1 = y^2, \quad 7y^2 + 1 = z^2.$$

By inspection, one readily gets an integral solution $y = 3, z = 8$ for the second equation. Did such problems provide the original motivation for exploring a systematic method of determination of integer solutions to the general *varga prakṛti*? Possible. In his commentary (*Sundarī*) on the *Laghu-Bhāskarīya*, Udaydivākara (1073) prescribes Jayadeva's general method for solving $7y^2 + 1 = z^2$ ([Sh], p 4).

But the central problems in the vast canvas of ancient Indian astronomy do not seem to seriously involve the Pell's equation. It is true that the examples of Brahmagupta on *varga prakṛti* (as also on other topics in algebra) are put in the language of astronomy. Given the exalted status of astronomy, it was natural to put the illustrative examples in a glamorous astronomy framework for greater impact. But considering the prevailing interests in astronomy, it looks more likely that these examples were invented to illustrate the rules; rather than the rules having emerged from an attempt to solve those examples. Astronomy alone might not have provided sufficient impetus for the sustained effort that would have gone into the partial (and later the general) solution to such a difficult problem.

(ii) Rational Approximation to $\sqrt{D}$

Āryabhaṭa's method for computing $\sqrt{D}$, when D is a perfect square, is well-known ([DS], p 170–175) — a modified version of his algorithm is taught in school arithmetic. But right from the Vedic times, there have also been efforts to obtain convenient, reasonably accurate, rational approximations to $\sqrt{D}$ when D is not a perfect square. In the Śulba Sūtras (c. 800 BCE), the most ancient mathematics treatises available, $\frac{7}{5}$, $\frac{17}{12}$, $\frac{577}{408}$ have been used as approximations for $\sqrt{2}$ ([D3], p 202; [Di], p 341; [Sa], p 18). These three fractions may be interpreted as arising out of solutions of $2x^2 \pm 1 = y^2$; for $2.5^2 - 1 = 7^2$, $2.12^2 + 1 = 17^2$ and $2.408^2 + 1 = 577^2$. [21] In fact, they are respectively the third, fourth and eighth convergents of the simple continued fraction expansion of $\sqrt{2}$. [22]

It is possible that, at some stage, Indians noticed that a sufficiently large integer solution (a, b) of $Dx^2 + 1 = y^2$ (or even $Dx^2 \pm c = y^2$ where c is a relatively small positive integer) will give a good approximation $\frac{b}{a}$ for $\sqrt{D}$ and that this realisation triggered a quest not only for a method of solving the *varga-prakṛti* but also a device for generating arbitrarily large solutions from a given one. The hypothesis may have a pedagogic usefulness, but there does not seem to be any concrete evidence so far to indicate that this was indeed the original motivation for Brahmagupta. Nārāyaṇa, who explicitly used the *varga-prakṛti* for approximating $\sqrt{D}$

[21] One may recall here that Archimedes (287-212 BCE) gave the approximations $\frac{265}{153}$ and $\frac{1351}{780}$ for $\sqrt{3}$ and that, in his commentary on Archimedes, Eutocius (700 CE) mentioned the relations $265^2 - 3 \times 153^2 = -2$ and $1351^2 - 3 \times 780^2 = 1$ as a verification of the validity of the approximations ([W2], p 16).

[22] To cite two other examples of rational approximations: Śrīdhara (750 CE), Āryabhaṭa II (950), Śrīpati (1039), Nārāyaṇa (1350) and Munīśvara (1603) gave the rule $\sqrt{D} \approx \frac{[\sqrt{DN^2}]}{N}$ for a conveniently chosen large integer N ([Bg], p 98–99). Jñānarāja (1503), his son Sūryadāsa (1541), and Kamalākara (1658) gave a method of successive approximation ([D2], p 193–194) which may be expressed as follows: Let $a_0 = a = [\sqrt{D}]$ and $a_{i+1} = \frac{1}{2}(a_i + \frac{D}{a_i})$; then $a_i \approx \sqrt{D}$ and each a_i is a better approximation than the previous a_{i-1}. The case $a_2 \approx \sqrt{D}$ was mentioned earlier in the ancient Bakhshālī Manuscript in the form $\sqrt{D} \approx a + \frac{r}{2a} - \frac{(\frac{r}{2a})^2}{2(a+\frac{r}{2a})}$, where $r = D - a^2$ ([D1], p 11–12). The Śulba fractions $\frac{17}{12}$ and $\frac{577}{408}$ are respectively the a_2 and a_3 for $D = 2$.

(see Section 4), lived more than seven centuries after Brahmagupta. It is however possible that Indians did have an early realisation of this application of large solutions of *varga-prakṛti* but did not mention it in any major text till the discovery of a general algorithm for finding a non-trivial solution of $Dx^2 + 1 = y^2$ for *any* D.

(iii) Pursuit as Pure Mathematics

The mathematical chapters of Brahmagupta's treatise ([B]) reveal the mind and spirit of a great pure mathematician. Many of Brahmagupta's discoveries show his enthusiastic pursuit of results from a feel for their intrinsic mathematical worth rather than the requirements of immediate applications. One may cite, for instance, his theorems (Ch 12, Verses 21, 28) on the area and diagonals of a cyclic quadrilateral (rediscovered by W. Snell in 1619) and his ingenious construction (Ch 12, Verse 38) of a cyclic quadrilateral whose sides, diagonals, circumradius and area are all rational and whose diagonals are perpendicular to each other (which drew the admiring attention of M. Chasles and E.E. Kummer in 19th century).

Now Āryabhaṭa had already presented a cryptic solution to the linear indeterminate equation $ax - by = c$. This problem probably arose from, and certainly had applications in, Indian astronomy. Subsequent algebraists, including Brahmagupta, brought clarity and made refinements on Āryabhaṭa's pioneering work. Having successfully dealt with the linear case, the pure mathematician's impulse might have driven Brahmagupta to take up the harder problem of the quadratic indeterminate equation. The peculiarity of the *varga-prakṛti* would have come out from his investigations. He would also have recognised the fundamental nature and importance of the equation $Dx^2 + 1 = y^2$ on which he clearly focussed considerable attention.

The achievements on Pell's equation tend to overshadow the fact that, at least from the time of Brahmagupta, Indian algebraists had produced a large bulk of work involving ingenious solutions of various types of indeterminate equations.[23] Most of these equations were clearly investigated for their own sake. This zest for exploration of higher degree indeterminate equations (after the application-oriented linear case) fits

[23]See [DS] for some examples.

into the general spirit of the era — the "Classical Age" of Indian history — when any department of knowledge or sphere of activity, once taken up, was pursued to its extreme.

Brahmagupta's penchant for pursuing problems in algebra for the sheer joy (and glory) is explicitly recorded towards the end of his chapter on algebra, i.e., Kuṭṭāka ([B], Chapter 18, Verses 99–100; [C], p 377):

sukhamātramamī praśnāḥ praśnānyātsahasraśaḥ kuryāt
anyairdattātpraśnānuktairvā sādhayetkaraṇaiḥ

jana saṁsadi daivavidāṁ tejo nāśayati bhānuriva bhānām
kuṭṭākārapraśnaiḥ paṭhitairapi kiṁ punarjñātaiḥ

"These questions are stated simply for delight. One may devise a thousand others; or may resolve the problems proposed by others, by the rules given here.

"As the sun eclipses the stars by his brilliance, so will an expert eclipse the glory of other astronomers[24] in assemblies of people, by the recital of algebraic problems, and still more by their solution."

While stating concrete examples, especially those involving the *varga prakṛti*, he often used the phrase *kurvannāvatsarād gaṇakaḥ* — "One who can solve it within a year (is truly a) mathematician." Clearly he revelled in the challenge posed by the *varga prakṛti*, the determination of a solution of which was described by Jayadeva ([Sh], p 14–15) as being "as difficult as setting a fly against the wind":

prakaṭitamatigahanamidaṁ marutimukhe makṣikākaraṇam

[24] *daivavid* means "destiny-knowing" or "astrologer" ([Mw], p 497). In this context, "astronomer" seems more appropriate. As some of the great mathematical astronomers (like Varāhamihira) were also famous for their knowledge of astrology, a sharp distinction was not always made. In the quoted line, the word *daivavid* might have been preferred to a standard term for astronomer for greater impact. Apart from the nuance of "godly" in *daiva*, the word seems to create a powerful sound-effect in that line.

8 Historical Context

In the history of indeterminate equations in India, the three greatest achievements are the *kuṭṭaka*, the *bhāvanā* and the *cakravāla*. The *kuṭṭaka* algorithm occurs in Āryabhaṭīya (499 CE). In the absence[25] of evidence to the contrary, one may regard Āryabhaṭa as the pioneer of *kuṭṭaka*.

Brahmagupta (628 CE) flourished a century after Āryabhaṭa. A truly original idea, the *bhāvanā* has the unmistakable stamp of Brahmagupta's genius. One does not see any seed of the *bhāvanā* in the treatises of his predecessors Āryabhaṭa and Bhāskara I or in the Bakhṣālī manuscript. None of the results in these remarkable texts come close to the algebraic sophistication shown by Brahmagupta's research on Pell's equation. In fact, in several results of Brahmagupta, one discerns a similar abrupt jump in the mathematical maturity level.

Brahmagupta was born in an era when symbolic algebra was at its infancy. He himself established some of its seminal features — he introduced the concept of zero as an integer in algebra, formulated the rules of arithmetic operations involving negative numbers and zero, contributed to the evolution of convenient notations and terminology (like the use of distinct letters for several unknowns) and the formation and handling of equations; and so on. It is astonishing that an ancient scientist, who had to develop such basic principles in algebra, could have taken the leap into investigation of a topic like Pell's equation — and that too with the mind and approach of a modern algebraist.[26]

The *bhāvanā* influenced, directly and indirectly, subsequent research on indeterminate equations by Indian algebraists. For tracing the possible flow and development of ideas, it is desirable to have some clarity regarding the knowledge and achievements at various time-points. However, due to the absence of certain crucial historical documents, the

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[25] Hardly any complete post-Vedic mathematics treatise, prior to Āryabhaṭīya, has survived.

[26] The achievement appears all the more startling when contrasted with the general evolution of algebraic thought: till the 16th century CE, Arab and European mathematicians struggled with problems involving equations of the type $ax + b = c$ (a, b, c positive numbers) as shown by the prevalence of the cumbrous "rule of false position".

modern historian faces some difficulties.

Now some of the crucial unresolved questions are: Who invented the *cakravāla* and when? What was the progress in indeterminate equations during the intermediary period (between Brahmagupta and the invention of *cakravāla*) and how were the results presented? To what extent had the awareness of this discovery spread among the mathematicians in the course of time? One has to remember the caveat of Weil ([W1], p 234) that "one should not invariably assume a mathematician to be fully aware of the work of his predecessors".

Perhaps one has to await the discovery of the lost algebra treatises of Śrīdhara, Padmanābha, Jayadeva, Śrīpati and other algebraists who flourished during the 5 centuries between Brahmagupta (628 CE) and Bhāskara II (1150 CE), or at least come across relevant quotations from these texts.

So far, the earliest author who refers to *cakravāla* is Udayadivākara (1073 CE) who quotes Jayadeva's verses on *cakravāla*. But we have no idea regarding the dates or other results of Jayadeva. Given the brilliance of the work, it is surprising that Jayadeva is not mentioned by Bhāskara II — he does not seem to have been referred to by most of the mathematicians.

The earliest extant original expositor[27] of the *cakravāla* (so far) is Bhāskara II (1150 CE) who attributes the name of the method (and hence the method) to earlier writers without specifying any name. Being a gifted algebraist himself, Bhāskara II would have undoubtedly realised the greatness of this work. Now, at the end of his algebra text *Bījagaṇita*, he makes a general acknowledgement of Brahmagupta, Śrīdhara and Padmanābha as his sources. There is then a strong possibility that Śrīdhara or Padmanābha invented, or at least wrote on, the *cakravāla*. But none of Padmanābha's works has been found. There is no clue as to his dates. Śrīdhara's date is estimated to be around 750 CE. But his algebra treatise is no longer extant. Some of its content, like the method of "completing the square" in quadratic equations, is known from references by Bhāskara. Another relevant missing text is the *Bījagaṇita* (algebra) of Śrīpati (dated 1039 CE).

[27] By original expositor we mean here one who is presenting the result in his own language and not quoting his predecessor(s).

If a book specialises on the most advanced topics of the time, or is too difficult, it is likely to be published in fewer numbers with fewer editions. Perhaps, for a similar reason, there was not enough circulation of the exclusively algebra treatises of the 8th–11th centuries.

Appendix: Original Verses on Brahmagupta's Bhāvanā

मूलंद्विधेष्टवर्गाद्गुणकगुणादिष्टयुतविहीनाच्च ।
आद्यवधोगुणकगुणःसहान्त्यघातेनकृतमन्त्यम् ॥
वज्रवधैक्यंप्रथमंप्रक्षेपःक्षेपवधतुल्यः ।
प्रक्षेपशोधकहृतेमूलेप्रक्षेपकेरूपे ॥

Theorem 1 (in Section 1) was stated by Brahmagupta in Verses 64–65 of Chapter 18 (titled Kuṭṭakādhyāyaḥ[28]) of Brāhma Sphuṭa Siddhānta ([B]) in the following form:

Atha Varga Prakṛtiḥ

mūlaṁ dvidheṣṭavargād guṇakaguṇādiṣṭa yutavihīnācca
ādyavadho guṇakaguṇaḥ sahāntyaghātena kṛtamantyam
vajravadhaikyaṁ prathamaṁ prakṣepaḥ kṣepavadhatulyaḥ

In symbols, the first line states:

$$[\text{If}] \quad y_i = \sqrt{Dx_i{}^2 + m_i}, \qquad i = 1, 2;$$

while the second and third line taken together mean:

[28] *Kuṭṭakādhyāyaḥ*=*Kuṭṭaka* (algebra) + *adhyāyaḥ* (chapter). The subject algebra was called *kuṭṭaka-gaṇita* before the use of the current Sanskrit term *bījagaṇita*. The term *kuṭṭaka* (pulverisation) indicates that it was regarded as the science of solving problems by a process of simplification — by breaking quantities (for instance, the coefficients or solutions of a given equation) into smaller pieces. The term, reminiscent of Fermat's descent, was particularly appropriate for the solution of the linear indeterminate equation pioneered by Āryabhaṭa ([Du1], p 10–22).

$$\text{[then]} \quad y = Dx_1x_2 \pm y_1y_2, \quad x = x_1y_2 \pm x_2y_1, \quad m = m_1m_2$$
$$\text{satisfy} \quad y^2 = Dx^2 + m.$$

(Mention of $y = Dx_1x_2 + y_1y_2$ and $x = x_1y_2 + x_2y_1$ is explicit; mention of $y = Dx_1x_2 \sim y_1y_2$ and $x = x_1y_2 \sim x_2y_1$ also seem implicit.)

A literal translation of Line 1 would be : "[Given] the square roots [y_i for $i = 1, 2$] of [the following] two [quantities]: the [two] desired squares [${x_i}^2$] multiplied by the guṇaka [D] and increased or decreased by the desired [c_i]." Thus $y_i = \sqrt{D{x_i}^2 \pm c_i}$, where c_i are positive integers; i.e., $y_i = \sqrt{D{x_i}^2 + m_i}$, where m_i are integers. Another literal translation could begin as follows: "[Consider following] two [integral/rational quantities] which have square-roots [i.e., are perfect squares ${y_i}^2$]: $\cdots$", i.e., ${y_i}^2 = D{x_i}^2 + m_i$ $(i = 1, 2)$.

A literal translation of Line 2: "The product of the [given] initial (i.e., first) roots [x_1, x_2] multiplied by the guṇaka [D] together with the product of the [given] final (i.e., second) roots [y_1, y_2] yields a [new] final (i.e., second) root [y] [of the equation $y^2 = Dx^2 + m$].", i.e., $y = Dx_1x_2 + y_1y_2$ or $y = Dx_1x_2 \pm y_1y_2$.

A literal translation of Line 3: "Cross-multiplication [yields new] first [root of the above equation]; [new] interpolator is equal to the product of the previous interpolators."

Theorem 1 was also stated in verse form by Ācārya Jayadeva, Bhāskara II (1150), Nārāyaṇa (1350), Jñanarāja (1503) and Kamalākara (1658).

Remark 1

As pointed out by Datta-Singh ([DS], p. 146), the word *dvidhā* in Line 1 has two-fold implication:

(i) Consider *two* triples satisfying $Dx^2 + m = y^2$.

(ii) Consider *two* copies of a triple satisfying $Dx^2 + m = y^2$.

Thus (ii) is a special case of (i) and, together with the next two lines, indicate Corollary 1: If $Dp^2+m=q^2$, then $D(2pq)^2+m^2=(Dp^2+q^2)^2$.

That the audience is also expected to read the implication (ii) in Line 1, and therefore Corollary 1 in Lines 1–3, is confirmed by the very next Line 4:

prakṣepaśodhakahṛte mūle prakṣepake rūpe

"On dividing the roots [obtained] by the [original] additive or subtractive, [roots for the] interpolator unity [will be found]." OR "On dividing [the roots obtained] by the square root of the [new] additive or subtractive, [roots for the] interpolator unity [will be found]."

Thus, we have, Corollary 2: If (p,q) is a root of $Dx^2 \pm c = y^2$, then $(\frac{2pq}{c}, \frac{Dp^2+q^2}{c})$ is a root of $Dx^2+1=y^2$.

Line 4 is in harmony with the previous lines if the implication (ii) is also considered; otherwise the language used might appear abrupt.

Remark 2

Both the additive and subtractive versions of Theorem 1 seem to find simultaneous mention in Brahmagupta's verses through the expressions *saha* and *aikyaṁ* with the ambiguous sense of "combining" Dx_1x_2 with y_1y_2 and x_1y_2 with x_2y_1. The word "combining" certainly includes the implication "adding". Thus the reference to the more important additive principle (producing new roots $Dx_1x_2+y_1y_2$ and $x_1y_2+x_2y_1$) is obvious. Presumably, "combining" also includes "taking difference"; the subtractive principle (generating $Dx_1x_2 \sim y_1y_2$ and $x_1y_2 \sim x_2y_1$) is unlikely to have been overlooked by Brahmagupta. (Also see Remark 4 below.) In any case, Jayadeva, Bhāskara II and subsequent writers clearly describe the two principles successively in separate verses.

Remark 3

Theorem 1 in the Introduction was stated in three equivalent forms. Brahmagupta's verses (quoted above) seem to resemble best the second form. But the perception of Theorem 1 in the first form (i.e., in terms of an operation $\odot$) comes out in the applications by Brahmagupta and others, in the version of Theorem 1 due to Bhāskara (quoted below) and, above all, in the choice and use of the term *bhāvanā* (composition).

Remark 4

During our literal translation, we had to insert several brackets '[ ]' to include expressions which are implicit in Brahmagupta's verses. The 3 lines, conveying Theorem 1, could appear obscure for a person not already familiar with the essence of the result. Cryptic brevity is a general feature of the original treatises of the earlier greats like Āryabhaṭa (499 CE) and Brahmagupta (especially the former). The verses were meant to indicate broad hints — not complete details. There used to be emphasis on the use of a learner's own intellect for filling up the details. Here one may point out that most modern mathematicians feel that a researcher eventually gains more insight into his area from a terse text than from a clearly spelt-out text, from an obscure important paper than from a lucid one.

Besides, conciseness was also a practical necessity. Palm-leaf manuscripts do not last long. Thus a treatise could be preserved only through memorisation or through repeated copying. In either case, the more concise the work, the better its chance of survival through faithful memorisation or copying. Brevity would have been particularly indispensable for an ancient author with enormous output. One can appreciate Brahmagupta's compulsions when one considers that, in spite of its oppressive terseness, the number of verses in Brāhma Sphuṭa Siddhānta exceeds a thousand.

With passage of time (and improvement of writing materials), the later treatises, even by stalwarts, give more and more elaborate versions of the results of the predecessors. Jayadeva's verses on the *bhāvanā* ([Sh], p. 4–8) have greater clarity than Brahmagupta's, Bhāskarācārya's version (quoted below) is still more lucid.

Detailed expositions on the works of the Masters were meant to be orally transmitted through the *guru-śiṣya* (mentor-disciple) link. The commentaries too try to clarify the implications of the passages of the original treatises. Brahmagupta's verses are explained by Pṛthūdaka Svāmi (860 CE); Bhāskara's verses by Kṛṣṇa (1600) and Sūryadāsa (1541).[29]

Brahmagupta's result, once stated, is not difficult to prove. Proofs

[29]The footnotes in [C], p 170–172 and 363–364 contain excerpts from these commentaries on Brahmagupta's and Bhāskara's verses describing the *bhāvanā*.

of such results are not to be expected in the original treatises (where even the statements of main results are terse). They were communicated orally and sometimes recorded in the later works. A proof of Theorem 1 is described by Kṛṣṇa in his *Bījapallava* (1600) — see ([DS], p 148–149).

The version of Bhāskara II

Bhāskarācārya's version ([Ba], p 22) of Brahmagupta's *bhāvanā* occurs in the section "Varga Prakṛti" of his text "Bījagaṇita" (Verses 70–71). We present below the transliteration followed by a translation.

iṣṭaṁ hrasvaṁ tasya vargaḥ prakṛtyā kṣuṇṇau yukto varjitā vā sa yena mūlaṁ dadyāt kṣepakaṁ taṁ dhanarṇaṁ mūlaṁ tacca jyeṣṭhamūlaṁ vadanti

hrasvajyeṣṭha kṣepakān nyasya teṣāṁ tānanyānvā'dho niveśya krameṇa sādhyānyebhyo bhāvanābhirbahūni mūlānyeṣāṁ bhāvanā procyate'taḥ

vajrābhyāsau jyeṣṭhalaghvostadaikyaṁ hrasvaṁ laghvorāhatiśca prakṛtyā kṣuṇṇā jyeṣṭhābhyāsa yug jyeṣṭhamūlaṁ tatrābhyāsaḥ kṣepayoḥ kṣepakaḥ syāt

hrasvaṁ vajrābhyāsayorantaraṁ vā laghvorghāto yaḥ prakṛtyā vinighnaḥ ghāto yaśca jyeṣṭhayostadviyogo jyeṣṭhaṁ kṣepo'trāpi ca kṣepaghātaḥ

"[Consider] the desired lesser root [x].[30] Its square, multiplied by *prakṛti* [D], is added or subtracted by some quantity [c] such that it [the sum or difference $Dx^2 \pm c$] gives a [integer] square root. The interpolator [$m = \pm c$], positive or negative, is called the *kṣepa*; the square root [y] is called the greater root.

"Set down successively the lesser root [x_1], the greater root [y_1] and the interpolator [m_1]. Place under them, the same or another [triple x_2, y_2, m_2], in the same order. From them, by repeated applications of the *bhāvanā*, numerous roots can be sought. Therefore, the *bhāvanā* is being expounded.[31]

[30] x and y in $Dx^2 + m = y^2$, which were called *ādya* (initial or first) and *antya* (final) respectively by Brahmagupta, were named *hrasva* (lesser) and *jyeṣṭha* (greater) respectively by Bhāskara II; the *guṇaka* D of Brahmagupta was called *prakṛti* by Bhāskara II.

[31] The last sentence comes from the phrase *bhāvanā procyate'taḥ* which can also

"[Consider] the two cross-products of the two greater and the two lesser roots. The sum $[x_1y_2 + x_2y_1]$ of the two cross-products is a lesser root. Add the product of the two [original] lesser roots multiplied by the *prakṛti* $[D]$ to the product of the two greater roots. The sum $[Dx_1x_2 + y_1y_2]$ will be a greater root. The product of the two [previous] interpolators will be the [new] interpolator.

"Again the difference between the two cross-products is a lesser root. The difference between the product of the two [original] lesser roots multiplied by the *prakṛti* and the product of the two greater roots will be a greater root. Here also, the interpolator is the product of the two [previous] interpolators."

We mention here that Bhāskarācārya referred to Brahmagupta as *gaṇaka cakra cūḍāmaṇi*[32] (jewel among mathematicians).

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be translated as "Therefore, it is called *bhāvanā*." Both meanings might have been intended. After all, Bhāskara II began the text with a deliberate pun on *sāṁkhyāḥ*, *satpuruṣa*, *bīja*, *vyakta*, etc!

[32] *gaṇaka*= mathematician, *cakra cūḍāmaṇi*= crest-jewel; round jewel in a coronet.

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The *Karaṇī*: How to Use Integers to Make Accurate Calculations on Square Roots

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Abstract

This paper presents the *karaṇī*, a mathematical construction to use integers to make calculations with square roots. Indian mathematicians invented new operations for this purpose (e.g. $(\sqrt{2}+\sqrt{8})^2 = 2+8+2\sqrt{2\times 8} = (\sqrt{18})^2$ for the sum of what they call the *karaṇī* 2 and 8, the sum of which is the *karaṇī* 18). This construction seems to be sophisticated, even useless, but we can find an explanation in a commentary (17th century): if all the calculations on square roots are made with *karaṇī*, and the approximate value is taken only at the end, the result is more accurate than if approximate values are taken at the beginning of the calculation.

This paper intends to present a mathematical construction made by Indian mathematicians in order to use integers in the calculations involving square roots: these mathematical objects are called *karaṇī*.

To begin with, it may be useful to introduce the authors and the texts on which this paper is based. According to the Indian tradition, knowledge develops and is transmitted through a fundamental text and the commentaries written on it. Here, the main text is the *Bījagaṇita* of Bhāskarācārya, and two commentaries are used: the *Sūryaprakāśa* by Sūryadāsa ([1]) and the *Bījapallava* by Kṛṣṇadaivajña ([2]).

There are two Indian mathematicians by the name of Bhāskara: one who lived in the 7th century and is often referred to as Bhāskara I and one who lived in the 12th century and is known as Bhāskarācārya (Bhāskara-ācārya: Bhāskara the master) or Bhāskara II; in this paper, his name will be shortened to Bhāskara because there is no ambiguity here: we shall never use the works of the first Bhāskara. In the same way, Kṛṣṇadaivajña, is abbreviated as Kṛṣṇa.

1 The author: Bhāskara

Bhāskarācārya was born in 1036 śaka, which is 1114 A.D., as he puts it himself at the end of the third part of his main work, the *Siddhānta-śiromaṇi*([3]): *"I was born in the year 1036 of* śaka *kings, during my thirty-sixth year I composed the* Siddhāntaśiromaṇi*".*[1]

Then, he gives the name of a town and its location, the name of the *gotra* he belongs to and the name of his father: *"There was at Vijjaḍaviḍa, a town located in the Sahya mountains... a twice-born from the Śāṇḍilya lineage... the virtuous Maheśvara..., born from the latter, the clever poet Bhāskara..."*[2]

What remains a mystery up to now is the exact name and location of the town: Vijjaḍaviḍa. The Sahya mountains are located in the northern part of the Maharashtra state and hold the well-known sites of Ajanta and Ellora; in these mountains, an inscription was discovered about 1850 in the basement of a temple at Pātnādevī, a small place near the modern Chalisgaon; this inscription gives Bhāskara's genealogy from the 10th to the 13th century, approximately, and was engraved to celebrate the foundation by his grandson of a school dedicated to the studies of Bhāskara's works in this very place. The name of Vijjaḍaviḍa is not quoted in this inscription which explains that king Jaitrapāla, from the Yādava dynasty, made Bhāskara's son, Lakṣmīdāsa, his astrologer and took him from *"this town"* to his capital; at that time, the capital of the Yādavas was Devagiri which has been identified as Daulatabad near

[1] rasaguṇapūrṇamahīsamaśakanṛpasamaye 'bhavan mamotpattiḥ । rasaguṇavarṣeṇa mayā siddhāntaśiromaṇiḥ racitaḥ ॥

[2] āsīt sahyakulācalāśritapure... vijjaḍaviḍe śāṇḍilyagotro dvijaḥ... maheśvarakṛtī... tajjas... suddhīḥ kavir bhāskaraḥ ।

Aurangabad and is not that far from Pātnādevī.

1.1 Bhāskara's works

The main work of Bhāskara is the *Siddhāntaśiromaṇi*; it is composed of four parts: the first two are mathematical ones, the last two are astronomical ones. The two mathematical parts are respectively entitled: *Līlāvatī* and *Bījagaṇita*.

Let us briefly explain the title *Siddhāntaśiromaṇi*: "The Diadem on the *Siddhānta*". *Siddhānta* means "settled opinion". In India this name was given to the fundamental astronomical works; there were five major *Siddhāntas* but there remains only one of them: the *Sūryasiddhānta*, "The *Siddhānta* of the Sun", thus called because it is assumed to have been revealed by the god Sun. The work of Bhāskara is based on it.

The *Līlāvatī* is a treatise of elementary calculus; numeration, operations, rules of proportions, calculation of areas, volumes and so on; it belongs to the class of *pāṭī-gaṇita* or *vyakta-gaṇita*. *pāṭī* means method, *gaṇita* means calculus and *vyakta* means manifested; this is a *"method of calculus"* or a *"calculus on manifested numbers"*.

We can find the name of the title at the end of the first stanza of the *Līlāvatī* which describes the method *(pāṭī)*: *"(...) I proclaim with soft and correct words, using short syllables, a method of good computation, which causes great satisfaction, which is clear and* ***possesses the grace of the play.****"*[3]

The Moghul Abū al Fayḍ Fayḍī reports a legend about this title, which is also a name given to a girl in India, in his translation of this work into Persian, in 1587. He says that Līlāvatī is the name of Bhāskara's daughter to whom this book was dedicated. We did not find anything about this in Sanskrit commentaries; commentators merely explain the formation of the word *līlā-vat*: "that which possesses the play, that which is like a game".

Bījagaṇita: this is the generic name for algebra. *Bīja* means seed, so *Bījagaṇita* is calculus on seeds, the seeds which potentially contain

[3](...) pāṭīṃ sadgaṇitasya vacmi caturaprītipradām prasphuṭām । saṃkṣiptākṣarakomalāmalapadair lālitya**līlāvatīm** ॥

calculus on manifested numbers; another name for this is *avyakta-gaṇita*: calculus on non-manifested numbers. We can find in the commentaries some parallel with the *vyakta* and *avyakta* worlds, the manifested and non-manifested worlds of the *Sāṃkhya* philosophy, the non-manifested world containing the manifested world before the creation.

The *Bījagaṇita* expounds calculus on negative and positive numbers, calculus on unknown quantities — these are the *avyakta*, the non-manifested numbers which contain the possibility of making "real computations" if you replace them by numbers (manifested-numbers); another name for unknown quantities is *varṇa*: color, letter — and it explains resolution methods of equations: linear, Diophantine, algebraic... The fourth chapter, following the chapter on unknown quantities, is devoted to the *karaṇī*.

The two astronomical parts of the *Siddhāntaśiromaṇi* are entitled: *Grahagaṇitādhyāya*, "Lesson on the Computation of Planets", and *Golādhyāya*, "Lesson on Spheres". This part contains some trigonometry: calculation of additions of sines etc.

The *Siddhāntaśiromaṇi* was written in verse, as is usually done for this kind of treatises, and it is often difficult to understand without the help of commentaries. Here, we shall use two commentaries on the *Bījagaṇita*: one is the *Sūryaprakāśa* by Sūryadāsa, composed around 1530; the other is the *Bījapallava* by Kṛṣṇadaivajña, composed around 1604.

The other Bhāskara's known works are: a commentary on his *Siddhāntaśiromaṇi*, the *Mitākṣara* "Having Measured Syllables" or *Vāsanābhāṣya*; these are the names given for small commentaries to explain works very briefly. The *Karaṇakutūhala*: "The Wonder of Astronomical Calculations" is a practical treatise on astronomy. It is dated 1183 by Bhāskara himself; this is the last date that is known about him and, for this reason, some people think that he died about 1185. We do not know exactly when and where. Some think that he was in charge of the astronomical observatory in Ujjain and that he died there, but there are no texts and no inscriptions to support these facts. The astronomical instruments that we can see nowadays in Ujjain were erected in the 18th century by the maharajah of Jaipur, Jai Singh II. The last work is a commentary on a mathematical work by Lalla, an astronomer from the 8th century: the *Śiṣyadhīvṛddhidatantra*: "A Treatise to Increase the Understanding of Students".

2 The commentators: Sūryadāsa and Kṛṣṇadaivajña

2.1 Sūryadāsa

We do not know much about Sūryadāsa. As is usually the case, he belonged to a family of astronomers and, very likely, lived in the western part of the Godāvarī valley during the first part of the 16th century. He is known for two commentaries: one on the *Līlāvatī*, entitled *Gaṇitāmṛtakūpikā*, "The Well of Nectar which is The Calculus", and one on the *Bījagaṇita*: the *Sūryaprakāśa*, "The Brightness of the Sun"; this title is a play on words: his own name, Sūryadāsa, but also the name of Bhāskara, which also means the Sun, as does the word *sūrya*. The name of his father is also known, Jñānarāja. He wrote an astronomical treatise too: the *Siddhāntasundaraprakṛti* or "The Charming Foundation of the Siddhānta", and a mathematical work, in imitation of Bhāskara's *Bījagaṇita*: the *Siddhāntasundarabīja*.

2.2 Kṛṣṇadaivajña

Kṛṣṇadaivajña was born in a family of astronomers who had settled in Varanasi at the end of the 16th century. He was a protégé of the Moghul emperor Jahāngir (1605-1627). His commentary on the *Bījagaṇita*, the *Bījapallava* or "The Sprout of the Seed", is dated Saturday, the fourth *tithi* of the dark fortnight of the *Caitra* month, in the year 1523 of the *śaka* era, namely: Tuesday, March 12th, 1602. He also wrote some examples *(udāharaṇa)* using the horoscopes he made about members of the Moghul dynasty and he may have composed a commentary on the *Līlāvatī*. Unlike the others, Bhāskara and Sūryadāsa, he seems to have had as a *guru*, not his own father, but a nephew of Gaṇeśa, another great Indian mathematician.

3 The karaṇī

3.1 The "concept" of *karaṇī*

The word *karaṇī* means producer. It is in the feminine for, in *Śulbasūtra* texts, "The Aphorisms on the Rope", it was originally in relation to the word *rajju*, or rope, which is feminine too. This *rajju karaṇī* has been used to produce geometric squares, that is a right angle; there are some aphorisms in the *Śulbasūtra* explaining how to place some marks on a rope, at the distance of 3, 4 and 5, in order to produce a right-angled triangle and hence a square. So, the word *karaṇī* was used to designate the side of a square and because of that, it can mean the square root of a number. From the 5th century before the common era to the 17th century, the word *karaṇī* seems to have had many meanings related to squares and square roots and consequently it is not easy to translate. Moreover, the mathematical concept expressed by this word in this papaer, has no equivalent in modern mathematics. For this reason we will not translate it. One common translation into English is surd. The word surd is ambiguous, for it means either irrational number or square root; there is no idea of irrational numbers in *karaṇī*, no idea of a number which cannot be expressed by a ratio, by a fraction of integers. As for the meaning "square root", this is not a correct translation for *karaṇī*, as *karaṇī* are not merely numbers, but numbers with a set of operations.

Bhāskara does not give any definition of *karaṇī*; the first rule, *"sūtra"* as this kind of stanza is called in Sanskrit, deals straight with the rules of addition and multiplication, but we can find some information in the commentaries. Here is what Sūryadāsa says about this first *sūtra*[4]:

"Now, [the author] who examines the nature of these karaṇīs*, under the pretext of explaining the rule of multiplication, says:* "vargeṇa"*.*

One will multiply a square ***vargeṇa*** *by a square number; likewise, one will divide a square by a square only; on the contrary, one will not multiply or divide a square by a number. It is pointed out by this, that what is of the nature of* karaṇī *by name is of the nature of a number*

[4]We will see the *sūtra* itself later on. The transliteration of all quoted texts is given at the end of the article, see page 133.

accepted with the quality of square; this is said by Nārayaṇa:

> *"The name of* karaṇī *will be for that [number] the square root of which must be taken.*[5]*"*

We shall examine the first part of this text later, and use now only the underlined part. What is important here is that if you consider the number two as a *karaṇī*, you must accept that this number is a square; this is reinforced by the above quotation from Nārāyaṇa who is a mathematician from the 14th century. A *karaṇī* is a number for which you have to compute the square root, so you must think that it is a square, you must think of a mental squaring operation.

Here is now what Kṛṣṇa says as an introduction to his commentary on this chapter:

"Now are commented the six operations on the karaṇī. *In this matter, one should understand that the six operations on* karaṇī, *are six operations done through the squares of two square root quantities. Because the origin of these six operations is preceded by a state of square, therefore, [there will be] also the use of the expression "state of* karaṇī*", in relation with these six operations, for a quantity producing a square root; this usage will not be possible if the calculation proceeds with the state of square root as the first step. The technical word "six operations on* karaṇī*" must be understood because of the necessity of such calculations about square root numbers; in this [case], that quantity for which a square root without a remainder is not possible, when its square root is required, is a* karaṇī, *but this is not merely a quantity which does not produce a square root; if it were so, there would always be the use of the expression "state of* karaṇī*" for two, three, five, six, and so on.*

— Let it be so!

No! If the operations were done on that basis, for instance, eight added to two would be eighteen, etc."

This commentary requires some explanations. Note that the *"six operations"* are the operations which are studied in the *Bījagaṇita*, whatever the objects (negative numbers, unknown quantities, *karaṇī*),

[5]Quoted by Sūryadāsa from Nārayaṇa's *Bījagaṇitāvataṃsa.*

namely: addition, subtraction, multiplication, division, squaring and square root. First, we recognise the same idea as in the commentary by Sūryadāsa: a *karaṇī* is a number which is the square of its square root. Of course, this is a truism. For a modern mathematician, every number is the square of its square root; but at the time of Bhāskara, and his commentators, it was not so obvious that a number which was not a square could have a square root; we can see that in this text: Kṛṣṇa speaks of a *quantity producing a square root*, the Sanskrit compound *mūla-da* (which gives a root) is used by Indian mathematicians to designate a perfect square. Here, we are told that a square can be called a *karaṇī* but — and it is the second fact to be noticed — *"in relation with these six operations"*, that is to say that we cannot separate, for a number, the state of being a *karaṇī* from the class of operations defined for them.

This becomes clearer with the end of the text: Kṛṣṇa explains that a *"quantity for which a square root without a remainder is not possible, when its square root is required, is a* karaṇī ***but*** *this is not merely a quantity which does not produce a square root".* Indeed! For if we keep in mind that every number which is a square could be called a *karaṇī*, every integer should be a *karaṇī*. The explanation comes right after, taking the form of a small dialogue, as is often the case in Sanskrit commentaries: an opponent argues *"Let it be so!"*: and what if we take this definition for granted? Kṛṣṇa answers: *"If the operations were to be done on that basis, for instance, eight added to two would be eighteen."* So we cannot call a number a *karaṇī* without thinking of how to make operations with it.

3.2 Rules of computation

We can now analyse the text of Bhāskara and see why *"eight added to two would be eighteen"*; here are the two first rules of this chapter:

> *Let us fix the* mahatī *as the sum of the two* karaṇīs *and the* laghu *as twice the square root of their product, the sum and the difference of these two are as for the integers. One will multiply and divide a square by a square. –1– But the square root of the larger [number] divided by the smaller [number],*

> *plus one or minus one, multiplied by itself and by the smaller [number], will also be respectively, the sum and the difference of these two* [karaṇīs]. *One will leave them apart if there is no square root. –2–*[6]

These *sūtras* give two ways of calculation for the sum and the difference of two *karaṇīs*: in the first one, two new objects are defined: the *mahatī*, the greater, which is the sum of the two *karaṇīs*, that is to say the sum of the two integers which measure the two *karaṇīs*; the second object is the *laghu*, the smaller, which is twice the square root of the product of the same integers. The names chosen for these technical words are obvious, for the sum of two integers is always greater than twice the square root of their product. Once the *mahatī* and the *laghu* have been calculated, the two integers produced are added and the result is the *karaṇī* sum of the two *karaṇīs*, that is to say: the square of the sum of the square roots of the two integers measuring the *karaṇī*.

If the difference of the *mahatī* and the *laghu* is computed, the result is the difference of the two *karaṇīs*. There is a problem with this definition: the difference between two *karaṇīs* is the same whether you compute the difference between the greater one and the smaller one or the smaller minus the greater.

Before going deeper in the explanations of these rules, let us see the example given by Bhāskara, and its solution by Sūryadāsa.

> *Say the sum and the difference of the two* karaṇīs *measured by two and eight and by three and twenty-seven and after a long while, if you know the six operations on* karaṇī, *say, dear, [the sum and the difference] of these two measured by three and seven.*[7]

Now the commentary by Sūryadāsa:

[6] yogaṃ karaṇyor mahatīṃ prakalpya ghātasya mūlaṃ dviguṇaṃ laghuṃ ca| yogāntare rūpavad etayoḥ sto vargeṇa vargaṃ guṇayed bhajec ca ‖ laghvyā hṛtāyās tu padaṃ mahatyāḥ saikaṃ nirekaṃ svahataṃ laghughnam | yogāntare staḥ kramaśas tayor vā pṛthaksthitiḥ syād yadi nāsti mūlam ‖

[7] dvikāṣṭamityos tribhasaṃkhyayoś ca yogāntare brūhi pṛthak karaṇyoḥ | trisaptamityoś ca ciraṃ vicintya cet ṣaḍvidhaṃ vetsi sakhe karaṇyāḥ ‖

In this case, putting down **ka** *2* **ka** *8, by:* "Let us fix the *mahatī* as the sum of the two *karaṇīs*", *the mahatī is 10.*

Now, the product of these two karaṇīs *is 16, its square root 4, when multiplied by two, the* laghu *is produced: 8.*

"The sum and the difference of these two are as for the integers"*: 18 and 2; these two* karaṇī*:* **ka** *18 and* **ka** *2 are the sum and the difference.*

The solution of this exercise put forward by Bhāskara is clear. We can observe how the mathematicians proceeded to differentiate a simple number and a number which must be considered as a *karaṇī*: they wrote the first syllable of the word *karaṇī* before the number: ***ka*** 2 and this means that this "2" is the square of the square root of 2, so, in this addition we have to find out the square of the square root which is the sum of the square root of 2 and the square root of 8; with our modern notations, we have:

$$(\sqrt{2} + \sqrt{8})^2 = 2 + 8 + 2\sqrt{2 \times 8} = (\sqrt{18})^2$$

If we could mix our modern formalism with the ancient Indian one, we would write that the set of *karaṇī* is a set of numbers with this particular addition:

$$\boldsymbol{ka}\ a \boxplus \boldsymbol{ka}\ b = \boldsymbol{ka}\ (a + b + 2\sqrt{ab})$$

It is obvious that this addition (and subtraction) is only defined if the product of the two integers measuring the *karaṇī* is a square. What happens if this is not the case is expressed by the last sentence of Bhāskara's rule: *"One will leave them apart if there is no square root."* This situation is illustrated by the third example given: the sum of the *karaṇī* measured by three and seven is impossible because 21 is not a square, so the solution of this exercise is merely: ***ka*** 3 ***ka*** 7.

This creates an extension of the meaning of *karaṇī*. In Sanskrit texts there are some statements like: *"Let the* karaṇī *be measured by 2, 3 and 5"* as if a *karaṇī* was also a composition of several *karaṇīs* the sum of which is not possible but which can be used for other calculations.

The second way for adding, or subtracting, the *karaṇīs* given by Bhāskara (see page 122) is very simply derived from the first one and there is no need to comment upon it:

$$(\sqrt{8} \pm \sqrt{2})^2 = 2(\sqrt{\frac{8}{2}} \pm 1)^2 = \left\{ \begin{array}{c} 18 \\ 2 \end{array} \right.$$

Let us focus on the sentence of the rule: *"One will multiply and divide a square by a square."* We have seen how one of the commentators, Sūryadāsa, had explained it (page 120) in order to put forward the nature of square of the *karaṇī*.

Here is now what Kṛṣṇa says about it: "vargeṇa vargaṃ guṇayed bhajec ca"*: here is what is said: when you want to multiply* karaṇīs, *if there is the state of multiplicand or the state of multiplier for some integer — or if you want to divide* karaṇīs*, if there is the state of dividend or the state of divisor for some integer — then, having squared the integer, the multiplication and the division can be performed, because a* karaṇī *has the nature of square."*

Another idea springs from this commentary: how to "embed" the integers in the *karaṇī* set; if we want to make operations mixing integers and *karaṇī*, we have to square the integers in order to give them a state of *karaṇīs* and this will change the general rules given for the operations. For instance, the rule for squaring the integers given in the *Līlāvatī* uses the identity $(a+b)^2 = a^2+b^2+2\,ab$. So Kṛṣṇa explains afterwards: *"for the square also, the fulfilment is in like manner because it (the square) is a kind of multiplication according to its nature of product of two equal [numbers]. Or, according to the method stated for the manifested numbers:* "The square of the last [digit] must be placed and [the other digit must be] multiplied by the last one increased two times..."[8]*, there will be a fulfilment [of this method] for the squaring of the* karaṇī *also but, as has been said:* "One will multiply and divide a square by a square"*, when it is said:* "multiplied by the last increased two times"*, we must understand: "multiplied by the last increased four times."*

Let us now make a digression. Notice the extreme degree of conciseness of Sanskrit works such as the *Bījagaṇita*: in one single rule, Bhāskara describes four of the six operations: addition, subtraction, multiplication and squaring. Although he describes the addition and

[8]sthāpyo 'ntyavargo dviguṇāntyanighnāḥ । svasvopariṣṭāc ca tathāpare 'ṅkās tyaktvāntyam utsārya punaś ca rāśim — *Līlāvatī*

the subtraction, by giving two methods, because these operations differ from the same operations applied to numbers or unknown quantities, a simple sentence is enough for him to say how to handle the multiplication and consequently, the squaring.

This method of exposition is usual in the fundamental works of Sanskrit literature and the works of Bhāskara are of this kind; they were used for centuries as a basis for mathematical teaching: students learned them by heart, then their masters composed commentaries which became original lectures. In order that they may be easily remembered, they were composed in verse, *"using short syllables"*, as seen earlier (page 117), and any unnecessary rule was avoided. So, in this chapter on the *karaṇī*, there will not be any proper rule for multiplication: once it is understood — with the help of the commentator — that for any "simple" *karaṇī*[9] the multiplication is merely given by:

$$(\sqrt{2})^2\,(\sqrt{8})^2 = (\sqrt{2 \times 8})^2$$

the general rule for *karaṇī* measured by more than one integer has already been given in a preceding chapter, the chapter on unknown quantities:

> *One must think here, in like manner, of the rule of multiplication by parts stated for manifested numbers, in the case of the square of non-manifested numbers and in the case of the multiplication of* karaṇīs.[10]

And even here, the rule refers to another rule given in the *Līlāvatī* which describes the property of distributivity of the multiplication with respect to the addition — the "multiplication by parts":

> *(...) Or the multiplicand, equal in number to the number of parts of the multiplier, being placed under each of them, is multiplied by these parts and added up.*[11]

[9]Let us call "simple" a *karaṇī* measured by a single integer

[10]avyaktavargakaraṇīguṇanāsu cintyo vyaktoktakhaṇḍaguṇanāvidhir evam atra |

[11]guṇyas tv adho 'dho guṇakhaṇḍatulyas taiḥ khaṇḍakaiḥ saṃguṇito yuto vā |

In fact, it is clear from the text of the *Līlāvatī* and its commentaries that the "parts" in question are either the result of splitting an integer into two (or more) parts in order to perform a mental calculation or the digits of the number with their decimal place value; so, this last rule can be used to perform calculations on polynomial-like quantities: a number considered in such a way being merely a polynomial in the powers of ten.

Let us see now the example given by Bhāskara for the multiplication of the *karaṇī*:

> *Set the multiplier as the* karaṇī *counted by two, three and eight and the multiplicand as counted by [the karaṇī] three with the integer five; say the product quickly. Or the multiplier is the two* karaṇīs *measured by three and twelve less the integer five.*[12]

and the solution given by Sūryadāsa:

Here, the multiplier is: **ka** *2* **ka** *3* **ka** *8.*

In like manner, the multiplicand is counted by three with five units; in this multiplicand, there are: **ka** *3* **rū** *5.*

One notices an integer: after taking its square, the state of karaṇī *must be brought about because it has been said:* "One will multiply and divide a square by a square". *By so doing,* **ka** *3* **ka** *25 are produced.*

Now, according to the method of the rule: "One must think here, in like manner, of the rule of multiplication by parts stated for manifested numbers, in the case of the square of non-manifested numbers and in the case of the multiplication of *karaṇī*", *after multiplication,* **ka** *54* **ka** *450* **ka** *9* **ka** *75 are produced.*

The rule *"One will leave them apart if there is no square root"* is used in this example and we discover a new formalism to denote integers: ***rū***, first syllable of the Sanskrit word *rūpa*, the meaning of which is "unity", "integer". The commentator squares this integer to transform it into a *karaṇī* before he performs the multiplication.

[12] dvitryaṣṭasaṃkhyā guṇakaḥ karaṇyo guṇyas trisaṃkhyā ca sapañcarūpā | vadhaṃ pracakṣvāśu vipañcarūpe guṇo 'thavā tryarkamite karaṇyau ||

According to Bhāskara's rule for the multiplication "by parts" (see page 126), we could represent the way to do this multiplication with the following table:

ka 2	***ka*** 3	***ka*** 8
ka 3 ***ka*** 25	***ka*** 3 *ka* 25	***ka*** 3 ***ka*** 25
ka 6 ***ka*** 50	***ka*** 9 ***ka*** 75	***ka*** 24 ***ka*** 200
† ‡		† ‡

The multiplier is split in to three parts, as stated in Bhāskara's rule, and placed in the first row; then the multiplicand is put under each part of the multiplier and the multiplication is performed in each cell of the table, the results being written in the third row.

We have put an identical symbol under the *karaṇī* for which the addition is possible.

†: $6 + 24 + 2\sqrt{6 \times 24} = 54$
‡: $50 + 200 + 2\sqrt{50 \times 200} = 450$

Once the addition is done, the result is the one given by Sūryadāsa.

The second example raises a problem because there is no mathematical notation for addition or subtraction; the Indians have developed formalisms in some branches of knowledge like grammar and mathematics but there are no signs to denote the operations. In this chapter on the *karaṇī*, putting two *karaṇīs* side by side indicates that it is the sum of these two *karaṇī* which is considered. This is the meaning of: *"One will leave them apart if there is no square root"*. For the subtraction, the notation is almost the same because it has been explained at the beginning of the *Bījagaṇita* that a subtracted positive number becomes a negative number, therefore, subtracting a number is only adding its opposite. There is a sign to denote negative numbers: a dot is placed over them; applying this notation to the *karaṇī* leads to:

$$\textbf{\textit{ka}}\ 8\ \textbf{\textit{ka}}\ \dot{2}$$

meaning that the *karaṇī* 2 is subtracted from the *karaṇī* 8.

As long as only *karaṇīs* are considered, no problems occur: the subtraction rule applies as it is formulated: the difference is the *karaṇī* measured by the number $8 + 2 - 2\sqrt{2 \times 8} = 2$. But in his example, Bhāskara says: *"(...) Or the multiplier is the two* karaṇī *measured by three and twelve less the integer five"* and according to the Indian notation system, we have to write:

$$\textbf{\textit{ka}}\ 3\ \textbf{\textit{ka}}\ 12\ \textbf{\textit{rū}}\ \dot{5}$$

Because *karaṇī* and integers are mixed in this multiplier, we have to square the integer in order to transform it into a *karaṇī* and when doing this, we will loose the "negative sign" showing that the last component must be subtracted.

To solve this problem, Bhāskara introduced a restriction to the general rule which says that the square of a negative quantity is positive:

> *The square of negative integers will also be negative if it is calculated for the reason of a state of* karaṇī. *Likewise, the square root of a* karaṇī *the nature of which is negative will be negative for the reason of creation of a state of integer.*[13]

With this rule, the multiplier becomes: ***ka*** 3 ***ka*** 12 ***ka*** $\dot{25}$, which can be simplified as ***ka*** 27 ***ka*** $\dot{25}$, by the addition of the *karaṇīs* 3 and 12 ($3+12+2\sqrt{3 \times 12} = 27$). Now, the multiplication can be performed in the same way as in the first example; let us summarize this with a table:

ka $\dot{25}$	***ka*** 27
ka 3 ***ka*** 25	***ka*** 3 ***ka*** 25
ka $\dot{75}$ ***ka*** $\dot{625}$	***ka*** 81 ***ka*** 675

Depending on the commentator, the result can be simplified in more than one way: noticing that 81 and 625 are squares, we can give them back their state of integers and subtract one from the other, because the square root of the "negative square", $\dot{625}$, remains negative according to the last given rule; we get ***rū*** $\dot{16}$. The two remaining *karaṇīs*, ***ka*** $\dot{75}$

[13] kṣayo bhavec ca kṣayarūpavargaś cet sādhyate 'sau karaṇītvahetoḥ । ṛṇātmikāyāś ca tathā karaṇyā mūlaṃ kṣayo rūpavidhānahetoḥ ॥

and ***ka*** 675 can be subtracted, for $675 \times 75 = 50625$ is a square and we get ***ka*** 300.

It may seem strange for a modern mathematician to state a general rule, such as: *"the square of a positive or a negative number is a positive number"*, then to restrict its range of application by another rule which may even contradict the general one. Nevertheless this is found very often in mathematical Sanskrit texts because the paradigm of logic in Sanskrit scientific knowledge is grammar rather than mathematics. We can see an example of this here and this procedure repeatedly occurs in Pāṇini's grammar which is the fundamental text of Indian scientific tradition.

There are two more operations to complete the six operations described in the *Bījagaṇita*: division and square root.

Division is easy to perform — and Bhāskara does not give any rule for this, only examples — because the algorithm given for the multiplication is the same as the one given for the unknown quantities and thus, as is shown by the two preceding tables, the division is very similar to today's Euclidean division of polynomials: it is sufficient to read these tables in the reverse order, making the third row the dividend and the first one the divisor to find out that the middle row is the quotient of the division. All the examples given for the multiplication are used in this way to explain the division in both chapters: the one about unknown quantities and the present one on the *karaṇī*.

The square root is rather difficult and we shall not discuss it in this paper. Let us just say that its algorithm is based on the identity:

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc$$

which the mathematicians inverted in order to find out the quantities a, b, c from the left member of the identity. We have just written down three quantities, but there are some examples given by Bhāskara with more than three *karaṇīs*.

3.3 The use of the *karaṇī*

What was the purpose of Indian mathematicians when they constructed these *karaṇīs*? They knew perfectly well how to calculate the approximate values of a square root $\sqrt{A}$, using the first two or three values of the sequence:

$$a_{n+1} = \frac{1}{2}(a_n + \frac{A}{a_n})$$

Let us consider what Kṛṣṇa says about it (this text is the paragraph that follows the text quoted on page 121):

"But it may be argued that these are only words! Why, then, trouble yourself to study these operations on karaṇīs *for, in common practice, there is no use of* karaṇīs *but only of the approximate values of their square roots and, with the use of six operations on numbers, these six operations [on* karaṇīs*] are meaningless. Moreover, even if the calculation with* karaṇīs *is done, in common use, [calculation] with approximate square roots from the beginning is better than this and is preferable to it.*

— This is not correct. If a rough square root is taken from the beginning, there will be a big roughness in its multiplication and so on; but if the calculation of karaṇīs, *which is minute, is performed, later, when the approximate square root is taken, there will be some difference but not very much; for this great distinction, the six operations with* karaṇīs *must necessarily be undertaken."*

As already seen this justification for the construction of *karaṇī* takes the form of a dialogue; an opponent develops the idea that this construction is useless and that only approximate calculations of square roots is enough for everyday transactions. He is told in return that if many operations are done with approximate values of square roots the final error is much bigger than if the calculations were done through the sophisticated construction of *karaṇī* and if the approximate value of the result is taken at the very end of the calculation.

The next paragraph justifies the location of this chapter in the complete book of Bhāskara's mathematical works; in Sanskrit commentaries, it is mandatory to give the reasons why a particular subject is studied and to justify its place in the succession of the topics developed by an author.

"Although it is suitable that these [six operations on karaṇī*] should be undertaken before the six operations on* varṇa[14]*, because they are closer to the operations on manifested numbers, according to the maxim of the needle and the kettle*[15]*, it is however suitable to undertake them immediately after the operations on* varṇa *for the reason that a great effort is required by their examination and understanding."*

References

[1] See, for instance, *Bījagaṇita* of Bhāskara with the commentary *Sūryaprakāśa* of Sūryadāsa, vol. I, ed. Pushpa Kumari Jain, Gaekwad's Oriental Series, No.182, Vadodara, 2001.

[2] See, for instance, *Bījagaṇita* of Bhāskara with the commentary *Bījapallavam* of Kṛṣṇa Daivajña, ed. T.V. Radhakrishna Sastri, Tanjore Saraswathi Mahal Series No.78, Tanjore, 1958.

[3] See, for instance, *Siddhāntaśiromaṇi* of Bhāskara with *Vāsanābhāṣya* and *Vāsanāvārttika*, ed. Muralidhara Chaturveda, Varanasi, 1981.

[14] Unknown quantities.

[15] This maxim is used to show that when two things — an easy one and a difficult one — must be done, the easier one should be first attended to just like when you have to prepare a needle and a kettle, you should first take up the preparation of the needle as it is an easier work compared to the preparation of a kettle. See Apte's Practical Sanskrit-English Dictionary.

Appendix: Sanskrit texts

Sūryadāsa, page 120:

athaitasyāḥ karaṇyā guṇanavidhikathanavyājena svarūpaṃ nirūpayann āha *vargeṇe*ti—

vargeṇa vargāṅkena vargaṃ guṇayet tathā vargeṇaiva vargaṃ bhajen na paraṃ tu rūpeṇa vargaṃ guṇayed bhajed vety arthaḥ। anena karaṇītvaṃ nāma vargatvenābhimatāṅkatvaṃ sūcitaṃ bhavati tad uktaṃ nārāyaṇena

mūlaṃ grāhyaṃ rāśer yasya tu karaṇīnāma tasya syāt।

iti

Kṛṣṇa, page 121:

atha karaṇīṣaḍvidhaṃ vyākhyāyate। atredam avagantavyaṃ mūlarāśyor vargadvārā yat ṣaḍvidhaṃ tat karaṇīṣaḍvidham iti। asya ṣaḍvidhasya vargatvapuraskāreṇaiva pravṛtter ata evāsmin ṣaḍvidhe mūladarāśāv api karaṇītvavyavahāraḥ karaṇītvapuraskāreṇa gaṇitapravṛttāv ayaṃ na syāt। karaṇīṣaḍvidham iti saṃjñā tu karaṇīrāśāv etasya gaṇitasyāvaśyakatvād draṣṭavyā। tatra yasya rāśer mūle 'pekṣite niragraṃ mūlaṃ na saṃbhavati sa karaṇī। na tv amūladarāśimātram। tathā sati dvitripañcaṣaḍādiṣu sarvadā karaṇītvavyavahāraḥ syāt।

— astu sa iti cet।

na। tathā sati tatprayuktaṃ kāryaṃ syāt। yathāṣṭau dvisaṃyutā aṣṭādaśaiva syur ity ādi॥

Kṛṣṇa page 125:

"vargeṇa vargaṃ guṇayed bhajec ca" iti। etad uktaṃ bhavati—karaṇīguṇane kartavye yadi rūpāṇāṃ guṇyatvaṃ guṇakatvaṃ vā syāt karaṇībhajane vā kartavye yadi rūpāṇāṃ bhājyatvaṃ bhājakatvaṃ vā syāt tadā rūpāṇāṃ vargaṃ kṛtvā guṇanabhajane kārye। karaṇyā vargarūpatvād iti।

Kṛṣṇa page 125:

vargasyāpi *samadvighāta*tayā guṇanaviśeṣatvād uktavat siddhiḥ। *"sthāpyo 'ntyavargo dviguṇāntyanighnā"* ityādinā vyaktoktaprakareṇa vā karaṇīvargasyāpi siddhiḥ syāt kiṃ tu *"vargeṇa vargaṃ guṇayed"* ityuktatvād *dviguṇāntyanighnā* ity atra caturguṇāntyanighnā iti draṣṭavyam ǀ

Kṛṣṇa page 131:

nanv astu paribhāṣāmātram idaṃ tathāpi kim anena karaṇīṣaḍvidhanirūpaṇaśrameṇa na hy asti loke karaṇībhir vyavahāraḥ kintu tadāsannamūlair eva tatṣaḍvidhaṃ ca rūpaṣaḍvidhenaiva gatārtham ǀ kiṃ ca kṛte 'pi karaṇīgaṇite tatas tadāsannamūlenaiva vyavahāraḥ tadvaraṃ prāg eva tadādara iti cet ǁ

maivam ǀ prāg eva sthūlamūlagrahaṇe tadguṇanādāv atisthūlatā syāt kṛte 'pi sūkṣme karaṇīgaṇite paścāt tadāsannamūlagrahaṇe kiṃ cid evāntaraṃ syān na mahad ity asti mahān viśeṣa iti karaṇīṣaḍvidham avaśyam ārambhanīyam ǀ

tad yady api vyaktaṣaḍvidhāntaraṅgatvād varṇaṣaḍvidhāt prāg evārabdhuṃ yuktaṃ tathāpy etasya nirūpaṇāvagamayoḥ prayāsagauravāt sūcīkaṭāhanyāyena varṇaṣaḍvidhānantaram ārambho yukta eva ǀ

Relations between Approximations to the Sine in Kerala Mathematics

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Abstract

The mathematicians of the school of Mādhava in late medieval Kerala (South India) described various series expressions for trigonometric quantities. This paper examines the reasoning underlying two such formulas and the possible connections between them.

1 Introduction

The school of Mādhava in fifteenth- and sixteenth-century Kerala is famous for its brilliant mathematical discoveries in a number of areas, including astronomical modeling, trigonometric series, infinitesimals, iterative functions, and solution of equations. Some of their best-known work involves expressions (attributed to Mādhava himself) for the Sine and Cosine that are equivalent to what we now call Taylor series expansions. The modern name can be misleading, however, since formulas that in modern mathematics represent essentially the same concept can differ widely in their original Sanskrit context. Exploring the apparent motivations of these rules, as suggested by their accompanying "yuktis" or rationales, provides a better understanding of their relationships and their underlying concepts of trigonometric series.

Most of the results discussed here date back to the time of Mādhava himself or his student Parameśvara, but the yuktis explaining them in detail did not appear, as far as the existing texts indicate, until about a century later. It is not known how they originated: they may have been written along with the original results in texts now lost, or passed down orally from teacher to pupil, or devised by later members of the school to make sense of the enigmatic formulas, or some combination of all of these.

2 The Sine and Cosine series in the sources

We begin our examination of the Kerala power series by quoting their formulations as they appear in Sanskrit sources, translating them as literally as possible into modern mathematical notation, and showing their relation to the modern forms of these series.

2.1 The "Mādhava-Newton" power series

The following statements about the Sine (here capitalized to indicate that it is scaled to a non-unity trigonometric radius R, instead of being scaled to 1 like the modern trigonometric functions) and Versine (the "versed sine", equal to the Radius minus the Cosine) are part of a yukti laid out by one Śaṅkara in the first half of the sixteenth century. They form part of his commentary *Yuktidīpikā* ("Lamp of Rationales") on an astronomical text called *Tantrasaṅgraha* ("Epitome of Astronomical Treatises") written in 1500 by his teacher Nīlakaṇṭha, who studied under the son of Mādhava's student Parameśvara. (See [12], 184–190, in which similar statements are attributed to a manuscript of the *Tantrasaṅgraha* itself, [7], 169–173, and [9].)

The yukti is derived from a Mālayālam exposition by Jyeṣṭhadeva, another of Śaṅkara's teachers, entitled *Yuktibhāṣa* or "Vernacular [Exposition] of Rationales". This part of Śaṅkara's commentary applies to the beginning of the second chapter, where the *Tantrasaṅgraha*, like other Sanskrit astronomical treatises, introduces trigonometry. Śaṅkara is explaining ([13], 118) a way to find accurate Sine and Versine values for a given arc θ (measured in arcminutes):

> [*Yuktidīpikā* 2, 440–441:] Having multiplied the arc and the results of each [multiplication] by the square of the arc, divide by the squares of the even [numbers] together with [their] roots, multiplied by the square of the Radius, in order. Having put down the arc and the results one below another, subtract going upwards. At the end is the Sine... ([3], 57)

> [*Yuktidīpikā* 2, 442–444:] Having multiplied unity and the results of each [multiplication] by the square of the arc, divide by the squares of the even [numbers] minus their roots, multiplied by the square of the Radius, in order. But divide the first [instead] by twice the Radius. Having put down the results one below another, subtract going upwards. At the end is the Versine...

In the computation for the Sine discussed in verses 440–441, the initial "result" is $\theta \cdot \theta^2$, the second $\theta \cdot \theta^2 \cdot \theta^2$, the third $\theta \cdot \theta^2 \cdot \theta^2 \cdot \theta^2$, and so forth. For $n = 1, 2, 3, \ldots$, the nth result is divided by R^2 times a term depending on the corresponding even number $2n$: namely, $(2n)^2 + 2n$. Thus the sequence of results will be

$$\begin{aligned} \frac{\theta^3}{R^2(2^2+2)} &= \frac{\theta^3}{R^2 \cdot 6}, \\ \frac{\theta^5}{R^4(2^2+2)(4^2+4)} &= \frac{\theta^5}{R^4 \cdot 120}, \\ \frac{\theta^7}{R^6(2^2+2)(4^2+4)(6^2+6)} &= \frac{\theta^7}{R^6 \cdot 5040}, \end{aligned} \tag{1}$$

and so forth. Since, for the nth such result, the denominator term

$$\prod_{i=1}^{n}((2i)^2 + 2i) = \prod_{i=1}^{n}(2i(2i+1)) = (2n+1)!, \tag{2}$$

it follows that Śaṅkara's sequence of recursive subtractions of terms from the next higher term will give

$$\operatorname{Sin}\theta = \theta - \left(\frac{\theta^3}{R^2 \cdot 3!} - \left(\frac{\theta^5}{R^4 \cdot 5!} - \left(\frac{\theta^7}{R^6 \cdot 7!} - \ldots\right)\right)\right). \tag{3}$$

The Versine computation in the subsequent verses similarly gives

$$\begin{aligned}\operatorname{Vers}\theta &= \frac{\theta^2}{R\cdot 2} - \left(\frac{\theta^4}{R^3\cdot 2(4^2-4)} - \left(\frac{\theta^6}{R^5\cdot 2(4^2-4)(6^2-6)} - \ldots\right)\right)\\ &= \frac{\theta^2}{R\cdot 2!} - \left(\frac{\theta^4}{R^3\cdot 4!} - \left(\frac{\theta^6}{R^5\cdot 6!} - \ldots\right)\right). \end{aligned} \tag{4}$$

Since Kerala trigonometry uses the standard value for R of about $3438 \approx 360\cdot 60/2\pi$, the division of $\theta^{(')}$ by R produces what we would call θ in radians. This means that the above expressions are equivalent to the following modern series normalized to $R = 1$:

$$\begin{aligned}\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\\ 1-\cos(x) &= \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots\end{aligned} \tag{5}$$

Other verses quoted by Śaṅkara in this yukti (*Yuktidīpikā* 2, 437–439: [13], 117–118), and elsewhere ascribed to Mādhava, contain numerical values for the coefficients (up to the fifth-order term) of a slightly different form of the same series for Sine and Versine. Hence it is clear that these power series were known (and probably originally developed) by Mādhava himself.

2.2 Mādhava and the "Taylor-series-like approximation"

Near the beginning of its second chapter, the *Tantrasaṅgraha* states a rule for determining the Sine of some arbitrary arc θ by means of Sines and Cosines from a Sine-table. More detailed analyses of this rule, and of an elaboration of it recorded by Mādhava's student Parameśvara, are given in [4], [5], and [10]. Briefly, choosing from the Sine-table the tabulated arc α closest to θ, and knowing the tabulated Sine and Cosine of α, we manipulate them as follows:

> [*Tantrasaṅgraha* 2, 10cd–12ab:] Having set down the two composite Sines [i.e., the tabulated Sine and Cosine of the tabulated arc] closest to the [arbitrary] arc whose Sine and Cosine are sought, one should compute the arc [of] deficiency

> or excess [depending on whether the desired Sine or Cosine is less or greater than the tabulated one].
>
> And set down as a divisor 13751 divided by twice the arcminutes of that [difference arc], for the purpose of the mutual correction of those [quantities]. Having first divided [either] one [of the tabulated Sine or Cosine by that divisor], add or subtract [the result] with respect to the other, according as the [difference] arc is excessive or deficient.
>
> Now in the same way, apply that [corrected quantity] times two to the other: this is the correction. ([13], 112)

The difference $(\theta - \alpha)$ is found in arcminutes, and used to compute a "divisor" D:

$$D = \frac{13751}{2(\theta - \alpha)} = \frac{2 \cdot 3437;45}{(\theta - \alpha)} \approx \frac{2R}{(\theta - \alpha)}. \tag{6}$$

The Sine and Cosine of α are "mutually corrected" after being divided by D, to give a first approximation to the Sine and Cosine of θ:

$$\operatorname{Sin}\theta \approx \operatorname{Sin}\alpha + \frac{\operatorname{Cos}\alpha}{D}, \qquad \operatorname{Cos}\theta \approx \operatorname{Cos}\alpha - \frac{\operatorname{Sin}\alpha}{D}. \tag{7}$$

The sign of the correction term depends on whether the function of θ is greater or less than that of α. These approximate results are then doubled, divided by D, and applied in another "mutual correction":

$$\begin{aligned} \operatorname{Sin}\theta &\approx \operatorname{Sin}\alpha + \left(\operatorname{Cos}\alpha - \frac{\operatorname{Sin}\alpha}{D}\right)\frac{2}{D}, \\ \operatorname{Cos}\theta &\approx \operatorname{Cos}\alpha - \left(\operatorname{Sin}\alpha + \frac{\operatorname{Cos}\alpha}{D}\right)\frac{2}{D}. \end{aligned} \tag{8}$$

Recalling that $D \approx 2R/(\theta - \alpha)$, and again normalizing to $R = 1$ as for modern trigonometric functions, we get equivalent expressions with x and a in the place of θ and α:

$$\begin{aligned} \sin(x) &\approx \sin(a) + \cos(a) \cdot (x - a) - \frac{\sin(a)}{2} \cdot (x - a)^2, \\ \cos(x) &\approx \cos(a) - \sin(a) \cdot (x - a) - \frac{\cos(a)}{2} \cdot (x - a)^2. \end{aligned} \tag{9}$$

Nīlakaṇṭha describes this method too as "spoken by Mādhava" ([13], 120).

2.3 The common equivalent in the modern Taylor series

Both sets of Mādhava's Sine and Cosine rules, as represented in equations (5) and (9), can be derived from the general Taylor polynomial

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots \quad (10)$$

where $f(x)$ is the sine or cosine function whose value at x is sought. Equation (9) gives the first three terms of the Taylor series, while equation (5) is equivalent to the corresponding Maclaurin series where $a = 0$.

The Taylor and Maclaurin expansions *per se*, of course, are theoretically and historically very closely intertwined. They emerged in the seventeenth and eighteenth centuries, in forms very similar to their modern representation, as part of the early calculus toolkit for dealing with arbitrary functions in the form of polynomials. The Maclaurin series was from the beginning considered as a special case of the Taylor series.

3 The series as explained in their yuktis

It is clear that the context in which Mādhava developed his "mutual corrections" of the Sine and Cosine or Sine and Versine was very different from the generalized techniques of successive differentiation employed by Taylor, Maclaurin, and their predecessors. What was the relation, if any, between Mādhava's two methods as perceived by the members of his school?

3.1 Correcting Sines by Versines

Śaṅkara's exposition culminating in the "Mādhava-Newton" series is detailed and intricate, comprising nearly a hundred Sanskrit verses ([13], 109–118). We can only touch upon its highlights here: more thorough summaries (though mostly without translations) of Jyeṣṭhadeva's rationale, from which Śaṅkara derived this explanation, may be found in [11], [7], [9], [12], and [14].

The foundation of the yukti is its division of a given arc into n unit arcs $\Delta\theta$. Each unit arc is bisected by a radius. For each successive arc

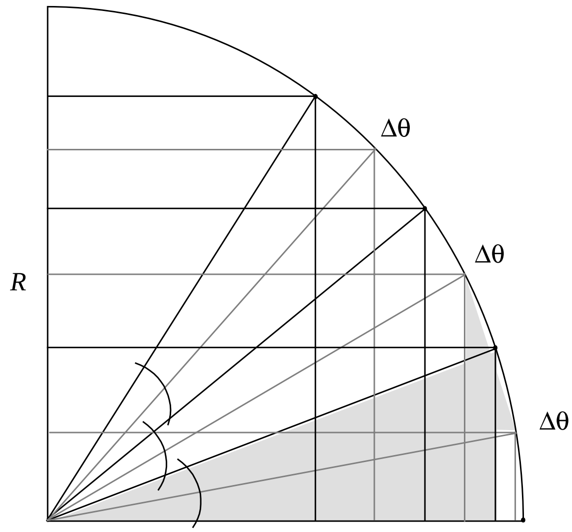

Figure 1: The quadrant with radius R, divided into unit arcs $\Delta\theta$

$\theta_i = i\Delta\theta$, the Sine and Cosine line segments Sin_i and Cos_i are drawn, as are the "medial" Sines and Cosines at the bisection points, $\text{Sin}_{i.5}$ and $\text{Cos}_{i.5}$.

As Figure 1 illustrates, the similar right triangles thus produced imply that the differences $\Delta\,\text{Sin}$ and $\Delta\,\text{Cos}$ between pairs of successive Sines and Cosines can be found from the medial Cosines and Sines. Each triangle with hypotenuse R and vertical leg $\text{Sin}\,\theta_i$ is similar to the smaller triangle with hypotenuse $\text{Crd}\,\Delta\theta$ and horizontal leg $\Delta\,\text{Cos}_{i.5}$, as in the pair of shaded triangles in the figure. As Śaṅkara puts it,

> [*Yuktidīpikā* 2, 349–352:] ... The change in the difference between Sines produced from the junctions of the corresponding arcs should be [proportional] to the change in the Cosine produced from the center of the arc. [Likewise,] the difference in Sine produced from the center of the arc is known as the change in Cosine-difference at the junctions of the corresponding arc. Thus the changes in difference are mutual[ly dependent].
>
> Then divide the Sine produced to the junction of the first arc,

> multiplied by the Chord, by the Radius: [it is] the difference of the Cosines in the centers of those arcs.

In our notation,

$$\Delta\,\mathrm{Cos}_{i+1} = \mathrm{Sin}_{i.5} \cdot \frac{\mathrm{Crd}}{R}, \quad \Delta\,\mathrm{Sin}_{i+1} = \mathrm{Cos}_{i.5} \cdot \frac{\mathrm{Crd}}{R};$$
$$\Delta\,\mathrm{Cos}_{i.5} = \mathrm{Sin}_{i} \cdot \frac{\mathrm{Crd}}{R}, \quad \Delta\,\mathrm{Sin}_{i.5} = \mathrm{Cos}_{i} \cdot \frac{\mathrm{Crd}}{R}. \tag{11}$$

For any consecutive pair of, e.g., Sine-differences, their "second difference" or "difference of the differences" $\Delta\Delta\,\mathrm{Sin}$ also depends on these quantities:

> [*Yuktidīpikā* 2, 352–354:] ... Having multiplied that [Cosine-difference, $\Delta\,\mathrm{Cos}_{1.5}$] by the Chord, divide again by the Radius. [That] should be the difference of the differences of the Sines produced to the junctions of the first and second [arcs]. Therefore, divide the first Sine, multiplied by the square of the Chord, by the square of the Radius. [The] quotient is the difference of the differences of the first and second [Sines].
>
> [*Yuktidīpikā* 2, 358–359:] ... In the same way, the division by the square of the Radius of every Sine multiplied by the square of the Chord should be similarly the difference of [its own difference and] the difference after that.

That is to say, in terms of the relations from equation 11,

$$\Delta\Delta\,\mathrm{Sin}_i = \Delta\,\mathrm{Sin}_i - \Delta\,\mathrm{Sin}_{i+1} = (\mathrm{Cos}_{i.5-1} - \mathrm{Cos}_{i.5}) \cdot \frac{\mathrm{Crd}}{R} = \Delta\,\mathrm{Cos}_{i.5} \cdot \frac{\mathrm{Crd}}{R}$$
$$= \mathrm{Sin}_i \cdot \frac{\mathrm{Crd}^2}{R^2}. \tag{12}$$

Consequently, any desired Sine-difference $\Delta\,\mathrm{Sin}_{i+1}$ (for $i > 0$) can be expressed in terms of the previous Sine-difference and the second-difference between them. But the previous Sine-difference itself can be expressed in terms of another second-difference, and so on, until we are left with a sum of second-differences. And each second-difference, we recall from equation (12), is dependent upon the corresponding Sine, as Śaṅkara goes on to state:

> [*Yuktidīpikā* 2, 361–363:] However many differences of [Sine-]differences there are, when the sum of that many, beginning with the first, is subtracted from the first difference, the desired difference is obtained. When it is desired to obtain the total of the differences of [Sine-]differences, [the quotient] from the sum of Sines multiplied by the square of the Chord, divided by the square of the Radius, should be the sum of the differences of the differences.

$$\begin{aligned}
\Delta\,\mathrm{Sin}_{i+1} &= \Delta\,\mathrm{Sin}_i - \Delta\Delta\,\mathrm{Sin}_i \\
&= \Delta\,\mathrm{Sin}_{i-1} - \Delta\Delta\,\mathrm{Sin}_{i-1} - \Delta\Delta\,\mathrm{Sin}_i \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \\
&= \Delta\,\mathrm{Sin}_1 - \Delta\Delta\,\mathrm{Sin}_1 - \ldots - \Delta\Delta\,\mathrm{Sin}_i \\
&= \Delta\,\mathrm{Sin}_1 - \sum_{k=1}^{i} \Delta\Delta\,\mathrm{Sin}_k \\
&= \Delta\,\mathrm{Sin}_1 - \sum_{k=1}^{i} \mathrm{Sin}_k \cdot \frac{\mathrm{Crd}^2}{R^2}. \qquad (13)
\end{aligned}$$

From equation (13) we can see that if we were to add up i consecutive Sine-differences, the result would involve a sum of sums of Sines (or, alternatively, of Cosine-differences, which are equal to Versine-differences). But the sum of i consecutive Sine-differences is also just the ith Sine itself:

> [*Yuktidīpikā* 2, 367–371:] Therefore, many sums of Sines one below another, ending with the sum of the first and second, are to be made here successively. [The quotient] from the sum of all those [sums], multiplied by the square of the Chord, divided by the square of the Radius—which is the result produced from the sums of the Sines—is equal to the quotient [from division] by the Radius of the sum of [the successive sums of] the Versine-differences, times the Chord.
>
> However many differences are considered in the case of [any] desired arc, subtract that [sum of sums] from the first difference times that number of differences. That should be

the Sine of the desired arc in the form of a sum of Sines of [arc-]portions.

$$\begin{aligned}\Delta\,\mathrm{Sin}_1 + \Delta\,\mathrm{Sin}_2 + \ldots + \Delta\,\mathrm{Sin}_i &= i\Delta\,\mathrm{Sin}_1 - \sum_{k=1}^{i-1}\sum_{j=1}^{k} \mathrm{Sin}_j \cdot \frac{\mathrm{Crd}^2}{R^2} \\ &= i\Delta\,\mathrm{Sin}_1 - \sum_{k=1}^{i-1}\sum_{j=1}^{k} \Delta\,\mathrm{Vers}_{j.5} \cdot \frac{\mathrm{Crd}}{R} \\ &= \mathrm{Sin}_i\,. \qquad (14)\end{aligned}$$

Śaṅkara now simplifies the above expression by taking $\Delta\,\mathrm{Sin}_1$ to be approximately equal to the unit arc $\Delta\theta$:

> [*Yuktidīpikā* 2, 375–376:] Because of the uniform smallness of the arc-portions, the first Sine-difference is assumed [equal to the unit arc]. That [$\Delta\,\mathrm{Sin}_1$] multiplied by the number of differences is the amount of the desired arc. Therefore, subtract from the desired arc the result produced as stated. The desired Sine remains there; it is [made] accurate by the rule to be stated [below].

So we can rewrite equation (14) as

$$\mathrm{Sin}_i \approx \theta_i - \sum_{k=1}^{i-1}\sum_{j=1}^{k} \mathrm{Sin}_j \cdot \frac{\mathrm{Crd}^2}{R^2} = \theta_i - \sum_{k=1}^{i-1}\sum_{j=1}^{k} \Delta\,\mathrm{Vers}_{j.5} \cdot \frac{\mathrm{Crd}}{R}. \qquad (15)$$

The process of "making the desired Sine accurate" commences with several more simplifying assumptions:

> [*Yuktidīpikā* 2, 379–382:] ...Whatever is the sum-of-sums of the arcs—owing to ignorance of [the values of] the Sines—is to be considered the sum [of the sums] of the Sines. But in that case, the last Sine should be the desired arc... Because of the smallness [of the unit], those arc-portions are considered [to be composed] with unity [i.e., as integers]. So the integers in the desired arc are equal to that [arc]. Therefore

> the sum of the Sines is assumed from the sum of the numbers having one as their first term and common difference. That, multiplied by the Chord [between] the arc-junctures, is divided by the Radius. The quotient should be the sum of the differences of the Cosines drawn to the centers of those arcs.
>
> [*Yuktidīpikā* 2, 384–385:] The other sum of Cosine-differences, [those] produced to the arc-junctures, [is] the Versine.[But] the two are approximately equal, considering the minuteness of the arc-division. On account of considering the arc-units [equal] to unity because of [their] minuteness, the Chord too is equal to that [unity], which makes no difference in the multiplier.

Śaṅkara points out, quite rightly, that we cannot use a sum of $i-1$ sums of unknown Sines to compute an equally unknown ith Sine. He proposes instead taking the sum of i sums of the successive arcs, which (if we take $\Delta\theta \approx \text{Crd}\,\Delta\theta \approx 1$) is just a sum-of-sums of successive integers, whose values are of course known. We will also assume that a sum of medial Cosine- (or Versine-) differences, which really equals a medial Versine, is roughly equal to the adjacent non-medial Versine. This gives us the following modified expression for Sin_i:

$$\text{Sin}_i \approx i - \sum_{k=1}^{i}\sum_{j=1}^{k} j \cdot \frac{1}{R^2} \approx i - \sum_{k=1}^{i}\sum_{j=1}^{k} \Delta\,\text{Vers}_j \cdot \frac{1}{R} \approx i - \sum_{k=1}^{i} \text{Vers}_k \cdot \frac{1}{R}. \quad (16)$$

A brief detour then provides a general expression for sums of successive integers and their successive sums:

> [*Yuktidīpikā* 2, 386–387:] Whatever is the product of however many numbers beginning with the first-term and increasing [successively] by one, that [product] is divided by the product of that many numbers beginning with one and increasing by one. The results one after another are the sums-of-sums of those [numbers].

That is, for a sequence of "however many" (say, q) consecutive integers beginning with any "first-term" p, Śaṅkara notes that

$$\frac{p \cdot (p+1) \cdots (p+q-1)}{1 \cdot 2 \cdots q} = \frac{(p+q-1)!}{(p-1)!q!} = \sum_{j_{q-1}=1}^{p} \cdots \sum_{j_2=1}^{j_3} \sum_{j_1=1}^{j_2} j_1. \quad (17)$$

Yet another simplifying assumption follows:

> [*Yuktidīpikā* 2, 389–393:] Half that square of the arc should be the sum of the [consecutive] arcs. Because half the product of the first-term and the first-term plus one is a sum: therefore, from the cube and squared-square [etc.] of the desired arc, divided by the product of numbers beginning with one and increasing by one, there are many resulting sums one after another.
>
> So in this case, the first sum of the arc-portions… should be considered the sum of the Sines. But that sum of Sines, multiplied by the Chord and divided by the Radius, is the desired Versine.

We know from the familiar rule for the sum of an arithmetic series that half the product $p(p+1)$ equals the sum of the first p integers. Śaṅkara now assumes that, since the desired arc $i \approx i+1$, the sum of the first i integers is approximately $i^2/2$, and generalizes that assumption (relying on an earlier demonstration in his commentary) to conclude that

$$\sum_{j_2=1}^{i} \sum_{j_1=1}^{j_2} j_1 = \frac{i(i+1)(i+2)}{1 \cdot 2 \cdot 3} \approx \frac{i^3}{6} \approx \sum_{k=1}^{i} \frac{k^2}{2},$$
$$\sum_{j_3=1}^{i} \sum_{j_2=1}^{j_3} \sum_{j_1=1}^{j_2} j_1 = \frac{i(i+1)(i+2)(i+3)}{1 \cdot 2 \cdot 3 \cdot 4} \approx \frac{i^4}{24} \approx \sum_{k=1}^{i} \frac{k^3}{6}, \quad (18)$$

etc. From this approximation and from equation (16), we can assume that

$$\sum_{k=1}^{i} k \cdot \frac{1}{R} \approx \mathrm{Vers}_i \approx \frac{i^2}{2R}. \quad (19)$$

We can also make a substitution for the double sum of integers in equation (16):

> [*Yuktidīpikā* 2, 401–403:] ...But the difference of the desired Sine and [its] arc is from the sum of the sum of the Sines. The square of the desired arc should be divided by the Radius; thence is [found] the Versine. But thence whatever [results] from the cube of the same desired arc [divided] by the square of the Radius, from that the quotient with six is approximately inferred [to be] the difference of the arc and Sine...

That is,

$$i - \mathrm{Sin}_i \approx \sum_{k=1}^{i} \sum_{j=1}^{k} j \cdot \frac{1}{R^2} \approx \frac{i^3}{6} \cdot \frac{1}{R^2}. \qquad (20)$$

Now Śaṅkara reminds us that this double sum of consecutive integers or arcs should really have been a double sum of Sines, so every arc in the sum needs to be corrected by the above expression for the difference between the arc and its Sine:

> [*Yuktidīpikā* 2, 406–408:] To remove the inaccuracy [resulting] from producing [the Sine and Versine expressions] from a sum of arcs [instead of Sines], in just this way one should determine the difference of the [other] Sines and [their] arcs, beginning with the next-to-last. And subtract that [difference each] from its arc: [those] are the Sines of each [arc]. Or else therefore, one should subtract the sum of the differences of the Sines and arcs from the sum of the arcs. Thence should be the sum of the Sines. From that, as before, determine the sum of the Versine-differences...
>
> [*Yuktidīpikā* 2, 417–419:] ...[W]hatever sum-of-sums is inferred from the determination of the Versine, the difference of the Sine and [its] arc is deduced from the sum-of-sums after that [one]. Thence in this case, the quotient from the product of the cube and square of the desired arc, [divided] by whatever is the product of five numbers beginning with one and increasing by one, [is] also divided by the squared-square of the Radius. The difference of the Sine and [its] arc determined in this way should be more accurate.

All the corrected consecutive terms must be re-summed to produce a more correct Versine, using the approximation for successive sums-of-sums in equation (18):

$$\mathrm{Vers}_i \approx \sum_{k=1}^{i}\left(k - \frac{k^3}{6R^2}\right)\cdot\frac{1}{R} = \sum_{k=1}^{i}\frac{k}{R} - \sum_{k=1}^{i}\frac{k^3}{6R^3} \approx \frac{i^2}{2R} - \frac{i^4}{24\cdot R^3}. \quad (21)$$

And we can use the same sort of term-by-term correction to modify all the Versines composing the Sine in equation (16):

$$\begin{aligned}\mathrm{Sin}_i &\approx i - \sum_{k=1}^{i}\mathrm{Vers}_k\cdot\frac{1}{R} \approx i - \sum_{k=1}^{i}\left(\frac{k^2}{2R} - \frac{k^4}{4!R^3}\right)\cdot\frac{1}{R}\\ &= i - \left(\sum_{k=1}^{i}\frac{k^2}{2R^2} - \sum_{k=1}^{i}\frac{k^4}{4!R^4}\right)\\ &\approx i - \frac{i^3}{3!R^2} + \frac{i^5}{5!R^4}. \end{aligned} \quad (22)$$

These recursive corrections of the Sine by the Versine and vice versa are then applied indefinitely to produce the general rules of equations (3) and (4).

3.2 Correcting Sines by Cosines

We turn now to the *Tantrasaṅgraha*'s "Taylor-series-like approximations" of equation (8) to see what we can glean of the reasoning behind them. Consider first of all that if we omit their final terms, we get expressions equivalent to the following:

$$\mathrm{Sin}\,\theta - \mathrm{Sin}\,\alpha \approx \mathrm{Cos}\,\alpha\cdot\frac{2}{D}, \qquad \mathrm{Cos}\,\alpha - \mathrm{Cos}\,\theta \approx \mathrm{Sin}\,\alpha\cdot\frac{2}{D}, \quad (23)$$

or equivalently

$$\frac{\theta-\alpha}{2R} \approx \frac{\mathrm{Sin}\,\theta - \mathrm{Sin}\,\alpha}{2\,\mathrm{Cos}\,\alpha} \approx \frac{\mathrm{Cos}\,\alpha - \mathrm{Cos}\,\theta}{2\,\mathrm{Sin}\,\alpha}. \quad (24)$$

As discussed in ([10], 287–288), Śaṅkara analyzes in a different commentary[1] on the same text (*Laghuvivṛti*, or "Minor Commentary") an

[1]Śaṅkara does not comment on this approximation in the *Yuktidīpikā*. I'm indebted to the reviewer of this paper for the important observation that such a yukti is in fact provided in the *Yuktibhāṣa* of Jyeṣṭhadeva.

approximation very close to the first of these, namely

$$\frac{\theta - \alpha}{2R} \approx \frac{\operatorname{Sin}\theta - \operatorname{Sin}\alpha}{\operatorname{Cos}\alpha + \operatorname{Cos}\theta}, \tag{25}$$

as follows:

> [*Laghuvivṛti* on *Tantrasaṅgraha* 2, 14cd–15ab:] Here, where the divisor should be made from the Cosine of the medial arc, it is said [to be made] with the sum of the Cosines of both full [arcs], by assuming that that [sum] equals twice the medial Cosine. But in reality, the sum of the Cosines of the two full [arcs] is somewhat less than twice the medial Cosine. Because of the deficiency of that divisor, the result of that is somewhat too big. But actually, that is what is desired: for that result is really the Chord, which is a little less than its arc...

In other words, if we considered the slightly smaller $\operatorname{Crd}(\theta - \alpha)$ rather than its arc, it would be more appropriate to write:

$$\frac{\operatorname{Crd}(\theta - \alpha)}{2R} \approx \frac{\operatorname{Sin}\theta - \operatorname{Sin}\alpha}{2\operatorname{Cos}\left(\alpha + \dfrac{\theta - \alpha}{2}\right)}. \tag{26}$$

But this is essentially the same similar-triangle relation expressed in equation (11): it states the difference between two successive Sines in terms of a linear proportion involving the Chord of the arc-difference, the Radius, and the medial Cosine. This exact linear proportion is what validates the slightly adjusted one, using the arc instead of the Chord, in equation (24). The "Mādhava-Newton series" and the "Taylor-series-like approximation" are therefore ultimately based on the same Rule of Three derived from similar right triangles.

Śaṅkara does not explain here the reason for adding the second-order correction term, but he proposes the option of including several more of them (see [10], 292):

> [*Laghuvivṛti* on *Tantrasaṅgraha* 2, 10cd–14ab:] Although here, prior to that, the quotient from half the Cosine with that same divisor [can] be applied to the Sine—and prior

> to that, the quotient-result from a fourth part of the Sine to the [half-]Cosine, and prior to that [the result] from an eighth part of the Cosine to the [fractional] Sine and [similarly, the result] from a sixteenth part of that [Sine] to the [fractional] Cosine—yet because of the smallness of that, it is to be considered negligible.

That is, the approximations in equation (8) might be expanded (if the size of the correction made it worthwhile) to

$$\begin{aligned}\operatorname{Sin}\theta \approx \operatorname{Sin}\alpha + \Bigl(\operatorname{Cos}\alpha - \Bigl(\operatorname{Sin}\alpha + \Bigl(\frac{\operatorname{Cos}\alpha}{2} - \Bigl(\frac{\operatorname{Sin}\alpha}{4} + \Bigl(\frac{\operatorname{Cos}\alpha}{8}\\ - \frac{\operatorname{Sin}\alpha}{16}\cdot\frac{1}{D}\Bigr)\frac{1}{D}\Bigr)\frac{1}{D}\Bigr)\frac{1}{D}\Bigr)\frac{1}{D}\Bigr)\frac{2}{D},\\ \operatorname{Cos}\theta \approx \operatorname{Cos}\alpha - \Bigl(\operatorname{Sin}\alpha + \Bigl(\operatorname{Cos}\alpha - \Bigl(\frac{\operatorname{Sin}\alpha}{2} + \Bigl(\frac{\operatorname{Cos}\alpha}{4} - \Bigl(\frac{\operatorname{Sin}\alpha}{8}\\ + \frac{\operatorname{Cos}\alpha}{16}\cdot\frac{1}{D}\Bigr)\frac{1}{D}\Bigr)\frac{1}{D}\Bigr)\frac{1}{D}\Bigr)\frac{1}{D}\Bigr)\frac{2}{D}.\end{aligned} \tag{27}$$

Recalling that $D = 2R/(\theta - \alpha)$, we may rewrite these as

$$\begin{aligned}\operatorname{Sin}\theta \approx \operatorname{Sin}\alpha + \operatorname{Cos}\alpha(\theta-\alpha) - \frac{\operatorname{Sin}\alpha(\theta-\alpha)^2}{2R^2} - \frac{\operatorname{Cos}\alpha(\theta-\alpha)^3}{8R^3}\\ + \frac{\operatorname{Sin}\alpha(\theta-\alpha)^4}{32R^4} + \frac{\operatorname{Cos}\alpha(\theta-\alpha)^5}{128R^5} - \frac{\operatorname{Sin}\alpha(\theta-\alpha)^6}{512R^6},\\ \operatorname{Cos}\theta \approx \operatorname{Cos}\alpha - \operatorname{Sin}\alpha(\theta-\alpha) - \frac{\operatorname{Cos}\alpha(\theta-\alpha)^2}{2R^2} + \frac{\operatorname{Sin}\alpha(\theta-\alpha)^3}{8R^3}\\ + \frac{\operatorname{Cos}\alpha(\theta-\alpha)^4}{32R^4} - \frac{\operatorname{Sin}\alpha(\theta-\alpha)^5}{128R^5} - \frac{\operatorname{Cos}\alpha(\theta-\alpha)^6}{512R^6}.\end{aligned} \tag{28}$$

And if we set $\alpha = 0$ in order to compare these approximations with their counterparts in equations (3) and (4), they reduce to

$$\begin{aligned}\operatorname{Sin}\theta \approx \theta - \frac{\theta^3}{8R^2} + \frac{\theta^5}{128R^4},\\ \operatorname{Cos}\theta \approx R - \frac{\theta^2}{2R} + \frac{\theta^4}{32R^3} - \frac{\theta^6}{512R^5}.\end{aligned} \tag{29}$$

3.3 Conclusion: the same or different series?

These commentaries of Śaṅkara contain almost the last known treatments within Mādhava's school of these two types of approximation to the Sine and Cosine. (The anonymous source in [3] appears to be somewhat later.) As understood in this mature form by Śaṅkara, the "Mādhava-Newton" series and the "Taylor-like" approximation are clearly independent rules. The former is valued largely for its production of Sine-values without the need for Sine-tables; the latter depends on a known tabulated Sine and Cosine to determine those of a nearby arc. Even if reduced to an equivalent form as in equation (29) (and there is no evidence that the "Taylor-like" approximation was ever actually handled this way in the Indian texts), they would be incompatible, since the integer coefficients in the denominators of the former are factorials, while those of the latter are powers of 2.[2]

Yet the two series may be linked conceptually. They are both founded on the Rule of Three arising from the same pair of similar right triangles in the subdivision of the quadrant. And they are both developed, after an initial approximation from this Rule of Three, via recursive mutual corrections of Sine-terms by Cosine-terms and vice versa. The power and fecundity of these basic concepts is illustrated in the variety of the approximations derived from them. In particular, the origin of these brilliant derivations in an elementary linear proportion recalls Bhāskara's comment on mathematical foundations: "Just as this universe is pervaded by Lord Nārāyaṇa (who removes the sufferings of those who worship him and is the sole generator of this universe), with his many forms—worlds and heavens and mountains and rivers and gods and men and demons and so on—in the same way, this whole type of computation is pervaded by the [rule of] three quantities."

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Algorithms in Indian Mathematics

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Abstract

Indian Mathematics is predominantly algorithmic. In fact, the very word "Algorithm" is derived from the name of Al Khwarizmi (c. 9^{th} Century) whose works played a crucial role in the transmission of Indian algorithmic procedures to the Islamic and later to the Western world. We shall discuss a few selected algorithms that are representative of the Indian mathematical tradition from the ancient *Śulbasūtrās* to the medieval texts of the Kerala School. In particular, we shall outline some of the constructions described in the *Śulbasūtras*, the algorithm for computing the cube-root given by Āryabhaṭa (c.499) and the *kuṭṭaka* and *cakravāla* algorithms for solving linear and quadratic indeterminate equations as discussed by Āryabhaṭa (c.499), Brahmagupta (c.628), Jayadeva (prior to the 11^{th} century) and Bhāskara (c.1150). We shall also discuss the efficient algorithms for accurate computation of π and the sine function due to Mādhava (c.14^{th} century) as discussed in the texts of the Kerala School of Mathematics and Astronomy.

1 Introduction

The very word *algorithm* is derived from the name of the famous ninth century Islamic mathematician Al-Khwarizmi who was greatly influenced by the Indian procedures in mathematics and wrote the famous book *Kitab al-hisab al-hindi* on the art of Hindu reckoning [1]. Indian Mathematics is predominantly algorithmic. It abounds in algorithms for quick and efficient computations with numbers, and algebraic, geometric and trigonometric quantities. However, it was understood that rationale and proofs have to be provided for all mathematical operations and they are to be found in the commentaries [2].

In this article, we describe some representative algorithms in Indian mathematics, as they have been developed over the ages. We begin with the *Śulbasūtras*, which are adjuncts to *Vedas* and contain the procedures for constructing various types of Vedic altars, of which we discuss some interesting examples. *Āryabhaṭīya* (c.499) is one of the earliest texts which gives algorithms for finding the square, cube, square root, cube root, etc. of a number. Detailed discussions of the same are found in the commentary on *Āryabhaṭīya* by Bhāskara I (c.629). We present the algorithms for finding the square and the cube root given in *Āryabhaṭīya* and its commentary. We also describe the *Kuṭṭaka* or the pulverizer algorithm for solving linear indeterminate equations.

We next discuss the solution of quadratic indeterminate equations or *Vargaprakṛti* which was first considered by Brahmagupta in his *Brāhmasphuṭa-siddhānta* (c.628). An optimal algorithm for the same called *Cakravāla* has been discussed by Bhāskara II in his *Bījagaṇita* (C.1150). The algorithm can at least be traced back to Jayadeva who lived prior to 11th century. We shall compare the *Cakravāla* algorithm with the one discovered by Wallis and Brouncker in the 17th century and proved later by Lagrange. We then discuss the algorithms for constructing the sine tables in Indian astronomy both in their earlier versions and as given in *Tantrasaṅgraha* (c.1500) and *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha Somayājī, the famous Kerala astronomer-mathematician. Finally, we describe the accurate calculation of π with a suitable remainder term, due to Mādhava (14th cent.) as described in the Kerala works *Yuktidīpikā* (c.1530), a commentary on *Tantrasaṅgraha* by Śaṅkara Vāriyar and *Yuktibhāṣā* (c.1530) of Jyeṣṭhadeva.

2 Geometrical construction in *Śulbasūtras* [3]

Śulbasūtras are part of *Kalpasūtras*, which are one of the six *Vedāṅgas*. These are essentially manuals for geometrical constructions of Vedic altars. Baudhāyana, Āpastamba, Kātyāyana etc. are some of the *Śulbasūtras*. Some of the simpler procedures discussed here are for dividing a line into equal parts, drawing perpendiculars, constructing squares and rectangles, etc. Pythagoras theorem or the 'Theorem of the square of the diagonal' plays a crucial rule in most of these constructions. This theorem was known in India, at least by the time of Baudhāyana *Śulbasūtra*, where right triangles with sides (3,4,5), (15,3,39), (7,24,25), etc., are mentioned. The earliest recorded explicit statement of the theorem is also to be found there.

> दीर्घचतुरश्रस्याक्ष्णया रज्जुः पार्श्वमानी तिर्यङ्मानी च यत् पृथग्भूते कुरुतः तमुभयं करोति ।
>
> (*Baudhāyana Śulbasūtra* 1.48)
>
> The diagonal cord of a rectangle makes both (the squares) that the vertical and horizontal sides make separately.

We give some illustrative examples of geometrical algorithms in the *Śulbasūtras* in the following.

2.1 To draw a square equal to the difference of two squares

> चतुरश्राच्चतुरश्रं निर्जिहीर्षन् यावन्निर्जिहीर्षेत् तस्य करण्या वृद्ध्रमुल्लिखेत् । वृद्ध्रस्य पार्श्वमानीं अक्ष्णया इतरत् पार्श्वम् उपसंहरेत् सा यत्र निपतेत्-तदपच्छिन्द्यात् ॥
>
> (*Āpastamba Śulbasūtra* 2.5)
>
> Wishing to deduct a square from a square one should cut off a segment by the side of the square to be removed. One of the lateral sides of the segment is drawn diagonally across to touch the other lateral side. The portion of the side beyond this point should be cut off.

$ABCD$ is the larger square. AE is the side of the smaller square. AD is drawn diagonally across with point A fixed till D touches EF at P. Considering the triangle AEP.

$$\begin{aligned} EP^2 &= AP^2 - AE^2 \\ &= AD^2 - AE^2 \end{aligned}$$

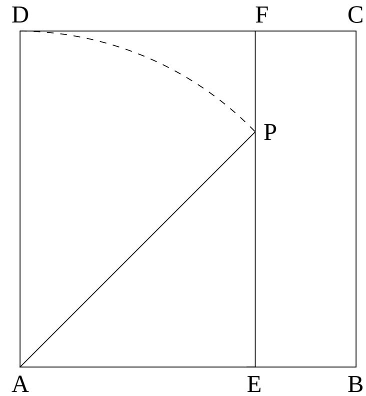

Fig.1

Hence, the square with side EP is the required answer.

2.2 To convert a rectangle into a square

> दीर्घचतुरश्रं समचतुरश्रं चिकीर्षन् तिर्यङ्मान्यापच्छिद्य शेषं विभज्योभयत उपदध्यात् । खण्डमागन्तुना संपूरयेत् । तस्य निर्ह्रास उक्तः ।
>
> (*Āpastamba Śulbasūtra* 2.7)
>
> Wishing to turn a rectangle into a square, one should cut off a part equal to the transverse side and the remainder should be divided into two and juxtaposed at the two sides (of the first segment). The bit (at the corner) should be filled by an imported bit. The removal of this has been explained already.

$ABCD$ is the given rectangle (see Fig.2). Consider the square AB_1C_1D with side AD. The remaining rectangle B_1BCC_1 is divided into two equal strips $B_1B_2EC_1$ and B_2BCE. The strip B_2BCE is cut off and applied to the side of the square DC_1. So the original rectangle has been converted into a square with side AD_1 with the small square C_1EC_2F unfilled at one corner. Hence,

$$\begin{aligned}\text{Area of the required square} &= \text{Area of the square } AB_2C_2D_1 \\ &- \text{Area of the square } C_1EC_2F.\end{aligned}$$

Using the previous procedure, this square is easily constructed.

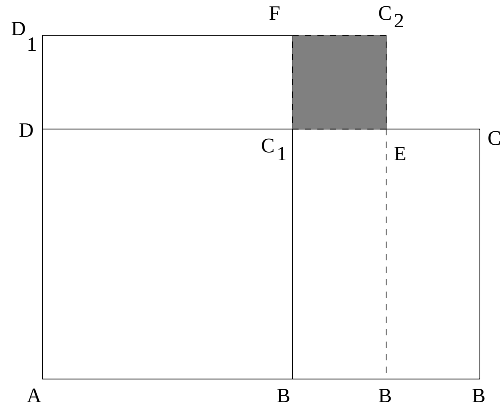

Fig.2

2.3 Combining equal squares

(a) Combining two equal squares

The rule in the *Āpastamba Śulbasūtra* (1.5) is

> समस्य द्विकरणी ।
>
> The diagonal of the square is the double-maker.

$ABCD$ is the given square with side a and area a^2. Then the square on the diagonal AC is the square with area $2a^2$. That is, $AC = \sqrt{2}a$. Incidentally, Fig.3 also indicates the proof.

Both *Āpastamba* (1.6) and *Baudhāyana Śulbasūtras* (1.62) give a very good approximation to the value of $\sqrt{2}$:

> प्रमाणं तृतीयेन वर्धयेत्तच्चतुर्थेन आत्मचतुस्त्रिंशोनेन सविशेषः ।
>
> The measure should be increased by one-third of itself, which is again increased by its one-forth and diminished by $\frac{1}{34}$ of that (second) increment. This is the *saviśeṣa*.

That is,

$$\sqrt{2} \simeq 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} = \frac{577}{408} \simeq 1.414216,$$

which is correct to 5 decimal places.

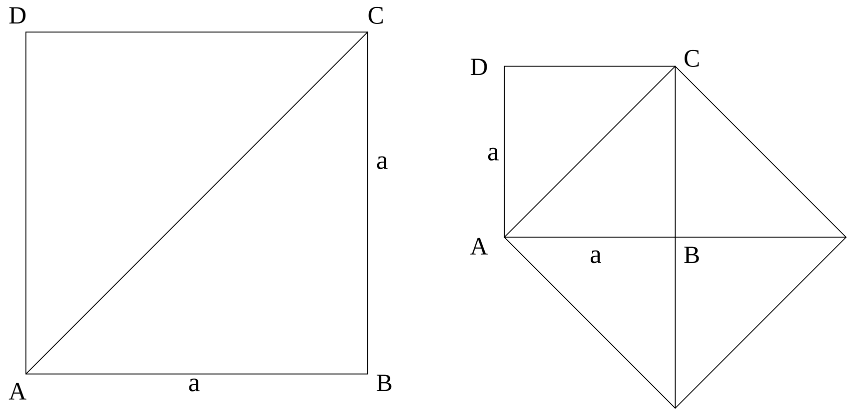

Fig.3

(b) An ingenious method for constructing a square with area n times a given square is given in *Kātyāyana-śulbasūtra* (6.7)

> यावत्प्रमाणानि समचतुरश्राण्येकीकर्तुं चिकीर्षेत् एकोनानि तानि भवन्ति तिर्यक् द्विगुणान्येकत एकाधिकानि । त्र्यस्रिर्भवति तस्येषुस्तत्करोति ॥
>
> As many sides (of equal side) as you wish to combine into one, the transverse line will be(equal to) one less than that; twice a side will be(equal to) one more than that. It will be a triangle. Its arrow(i.e., altitude) will do that.

Let the side of the original square be a. Construct an isosceles triangle ABC (see Fig.4) with base

$$BC = (n-1)a,$$

and sides

$$AB = AC = \frac{(n+1)a}{2}.$$

This is easily achieved by stretching the mid point A of a rope with length $(n+1)a$ away from BC such that the rope is taut. Then the altitude AD is the side of the square with area na^2, for,

$$\begin{aligned} AD^2 &= AB^2 - BD^2 \\ &= \left[\frac{(n+1)a}{2}\right]^2 - \left[\frac{(n-1)a}{2}\right]^2 \\ &= na^2. \end{aligned}$$

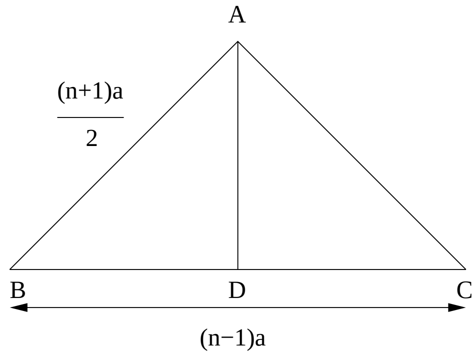

Fig.4

3 Algorithms for square and the cube root

3.1 Square

One of the earliest example of an algorithm in arithmetic is the one for squaring a number. In his *Āryabhaṭīya-bhāṣya* (c.629) Bhāskara I cites the following as an ancient rule for squaring [4]:

> अन्त्यपदस्य वर्गं कृत्वा द्विगुणं तदेवचान्त्यपदम् ।
> शेषपदैराहन्यात् उत्सार्योत्सार्य वर्गविधौ ॥

> To calculate the square of a number, the square of the last digit is to be placed over it. And the rest of the digits, doubled and multiplied by the last are to be placed over them respectively. Then, (omitting the last digit and) moving the rest by one place, the process is repeated.

Consider for example, the square of 125. The procedure is summarized below.

1	5	6	2	5	
				25	$5^2 = 25$
		4	20		$2^2 = 4,\ 5.2.2 = 20$
1	4	10			$1^2 = 1,\ 2.2.1 = 4,\ 5.2.1 = 10$
1	2	5			

It is noteworthy that this 'ancient rule' for squaring is already optimal. The mechanical method of squaring a n - digit number involves n^2 multiplications, whereas the above method clearly involves only $\frac{n(n+1)}{2}$ multiplications.

3.2 Cube root

An algorithm for square root is given in *Āryabhaṭīya* . Indeed, texts prior to this work also deal with square roots. However, the first clear enunciation of the cube root algorithm is found in *Āryabhaṭīya* , which we proceed to describe [5]

> अघनाद् भजेद् द्वितीयात् त्रिगुणेन घनस्य मूलवर्गेण ।
> वर्गस्त्रिपूर्वगुणितः शोध्यः प्रथमाद् घनश्च घनात् ॥
>
> (Having subtracted the greatest possible cube root from the last cube place and then having written down the cube root of the number subtracted in the line of the cube root), divide the second non-cube place (standing on the right of the last cube place) by thrice the square of the cube root (already obtained); (then) subtract from the first non-cube place (standing on the right of the second non-cube place) the square of the quotient multiplied by thrice the previous (cube root); and (then subtract) the cube (of quotient) from the cube place (standing on the right of the first non-cube place) (and write down the quotient on the right of the previous cube root in the line of the cube root, and treat this as the new cube root. Repeat the process if there are still digits on the right).

We consider the cube root of 17,71,561 as an example. Beginning from the units place, the notational places are called cube place (c), first non-cube place (n), second non-cube place(n'), cube place(c), first non-cube place(n), second non-cube place(n'), and so on.

		c	n'	n	c	n'	n	c	121
		1	7	7	1	5	6	1	(line of cube root)
Subtract 1^3		1							
Divide by 3.1^2	3)	0	7	(2					
		0	6						
			1	7					
Subtract $3.1.2^2$			1	2					
				5	1				
Subtract 2^3				0	8				
Divide by 3.12^2	432)			4	3	5	(1		
				4	3	2			
						3	6		
Subtract $3.12.1^2$						3	6		
							0	1	
Subtract 1^3								1	
								0	

The process ends and the cube root is 121. The algorithm is obviously based on the algebraic identity: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

4 *Kuṭṭaka* or 'pulveriser' algorithm for linear indeterminate equations

The subject of first order indeterminate equations was considered very important by the ancient Indian mathematicians and most of them have dealt with the *Kuṭṭaka* or pulveriser method to solve them, beginning with *Āryabhaṭīya* . They are important in astronomy also, for instance, in the calculation of *Ahargaṇa* (the number of days elapsed from a given epoch) from the mean longitudes of planets. This algorithm also plays a crucial role in the solution of the much more difficult second order indeterminate equations.

In one of its versions, the problem is to find an integer N which being divided by two given integers (a, b) will leave two given remainders (r_1, r_2). Thus we have

$$N = ax + r_1 = by + r_2.$$

Kuṭṭaka method for this equation is summarized in two verses in *Āryabhaṭīya* (*Gaṇitapāda*, 32, 33).

अधिकाग्रभागहारं छिन्द्यादूनाग्रभागहारेण ।
शेषपरस्परभक्तं मतिगुणमग्रान्तरे क्षिप्तम् ॥
अध उपरिगुणितमन्त्ययुगूनाग्रच्छेदभाजिते शेषम् ।
अधिकाग्रच्छेदगुणं द्विच्छेदाग्रमधिकाग्रयुतम् ॥

Below we give the translation of these verses by Datta and Singh following the interpretation of Bhāskara I [6]:

> Divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remainder. The remainder (and the divisor corresponding to the smaller remainder) being mutually divided, the last residue should be multiplied by such an optional integer that the product being added (in case the number of quotients of the mutual division is even) or subtracted (in case the number of quotients is odd) by the difference of the remainders (will be exactly divisible by the penultimate remainder. Place the quotients of the mutual division successively one below the other in a column, below them the optional multiplier and underneath it the quotient just obtained). Any number below (i.e., the penultimate) is multiplied by the one just above it and then added to the one just below it. Divide the last number (obtained by doing so repeatedly) by the divisor corresponding to the smaller remainder; then multiply the remainder by the divisor corresponding the smaller remainder and add the greater remainder. (The result will be) the number corresponding to the two divisors.

In the following we take $r_1 > r_2$, so that a is the divisor corresponding to the greater remainder, and b the one corresponding to the smaller remainder. Let

$$c = r_1 - r_2.$$

We write down the procedure, when the number of quotients (ignoring the first one q, as is usual with Āryabhaṭa) is even.

$$
\begin{array}{lllllll}
b) & a & (q & & & & \\
 & \underline{bq} & & & & & \\
 & r_1) & b & (q_1 & & & \\
 & & \underline{r_1 q_1} & & & & \\
 & & r_2) & r_1 & (q_2 & & \\
 & & & \underline{r_2 q_2} & & & \\
 & & & \cdot & & & \\
 & & & \cdot & & & \\
 & & & \cdot & & & \\
 & & & & r_{2n}) & r_{2n-1} & (q_{2n} \\
 & & & & & \underline{r_{2n} q_{2n}} & \\
 & & & & & r_{2n+1} &
\end{array}
$$

Now a number t (*mati*) is chosen such that $r_{2n+1}t + c$ is divisible by r_{2n} with quotient s. Then these are set down in the form of a *vallī* (column) and the successive columns are generated:

$$
\begin{array}{lllll}
q_1 & q_1 & q_1 & \cdots\cdots & q_1\beta_{2n-1} + \beta_{2n-2} \\
q_2 & q_2 & q_2 & \cdots\cdots & \beta_{2n-1} \\
\cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & & \\
q_{2n-1} & q_{2n-1} & q_{2n-1}\ \beta_1 + t = \beta_2 & & \\
q_{2n} & q_{2n}t + s = \beta_1 & & & \\
t & t & & & \\
s & & & &
\end{array}
$$

Divide $q_1\beta_{2n-1} + \beta_{2n-2}$ by b. The remainder is x and $N = ax + r_1$.

When we divide this by b, the quotient is y and the remainder is r_2. When the number of quotients (omitting q) is an odd integer $2n-1$, the number t is chosen such that $r_{2n}t - c$ is divisible by r_{2n-1}.

Example: To solve $45x + 7 = 29y$.

Here

$$a = 45,\ b = 29,\ r_1 = 7,\ r_2 = 0.$$

29)	45	(1				
	29					
	16)	29	(1			
		16				
		13)	16	(1		
			13			
			3)	13	(4	
				12		
				1		

Here the number of quotients (omitting the first) is odd. t should be chosen such that $1 \times t - 7$ is divisible by 3. Hence t is chosen to be 10. Therefore we have,

1	1	1	92
1	1	51	51
4	41	41	
10	10		
1			

Now, $92 = 29 \times 3 + 5$. Therefore,

$$N = 49 \times 5 + 7 = 29 \times 8.$$

Hence,

$$x = 5,\ y = 8.$$

5 *Varga-prakṛti*: Quadratic indeterminate equations

The quadratic indeterminate equation

$$x^2 - Dy^2 = 1,$$

for a non square integer D is generally referred to as Pell's equation, though the 17th century English mathematician Pell had very little to do with either posing the problem or solving it. The problem of finding the integer solutions x, y to this equation was posed as a challenge to the European mathematicians by Fermat in 1653, for specific values of $D = 61$, 109, 149, etc. The English mathematicians Brouncker and Wallis solved the equation. Fermat is credited with proving that the equation has infinite number of solutions.

In fact, the quadratic indeterminate equations of the form

$$x^2 - Dy^2 = K,$$

known as *Varga-prakṛti* had been considered nearly thousand years earlier by Brahmagupta in his *Brāhmasphuṭa-siddhānta* (c.628). D, the given non-square integer is called "*prakṛti*", K, a given integer, is called *Kṣepa* and the integer solutions x and y are called *Jyeṣṭha-mūla* and *Kaniṣṭha-mūla* respectively. The fact that the equation has infinite number of solutions is implied by Brahmagupta's *Bhāvanā* Principle, to be described below. The equation $x^2 - Dy^2 = 1$, had been solved for all D by Jayadeva (c.11th century or earlier) as cited by Udayadivākara (c.11th century) in his commentary *Sundarī* on *Laghu-bhāskarīya* of Bhāskara I [7]. Bhāskara II has also discussed this *Cakravāla* algorithm in his *Bījagaṇita* (c.1150) [8].

The motivation for solving such equations was probably to find rational approximation for surds. From $x^2 - Dy^2 = 1$, we find

$$\left|\sqrt{D} - \frac{x}{y}\right| \leq \frac{1}{2xy},$$

so that $\sqrt{D} \approx \frac{x}{y}$, if x and y are large. For instance,

$$577^2 - 2 \times 408^2 = 1,$$

so that

$$\sqrt{2} \approx \frac{577}{408}.$$

It is noteworthy that this is the value of $\sqrt{2}$ given in *Śulbasūtras*. The *Cakravāla* algorithm makes crucial use of the *Bhāvanā* Principle of Brahmagupta which is summarised below.[1]

[1]Further details regarding the *Bhāvanā* principle may be found in the contribution

5.1 Brahmagupta's *Bhāvanā*

If

$$x_1^2 - Dy_1^2 = K_1 \qquad \text{and} \qquad x_2^2 - Dy_2^2 = K_2,$$

then

$$(x_1x_2 \pm Dy_1y_2)^2 - D(x_1y_2 \pm x_2y_1)^2 = K_1K_2.$$

So, if the solution of the *Varga-prakṛti* for the *Kṣepas* K_1 and K_1 are known, the solution for the *kṣepa* K_1K_2 can be found immediately [9].

In particular,

$$x^2 - Dy^2 = 1 \qquad \Rightarrow \qquad (x^2 + Dy^2)^2 - D(2xy)^2 = 1$$

Thus, if one solution of the equation is known, an infinite number of solutions can be found.

Brahmagupta's *Bhāvanā* not only helps in finding any number of solutions from just one solution, it also enables us to solve the $K = 1$ case, provided we know a solution for $K = -1$, or ± 2 or ± 4. We give the solutions (x, y) of $x^2 - Dy^2 = 1$ in terms of the solutions (x_1, y_1) of $x_1^2 - Dy_1^2 = K$, when K assumes any of these five values below:

$$\begin{aligned}
K = -1 \ : \ x &= x_1^2 + Dy_1^2, \ \ y = 2x_1y_1. \\
K = \pm 2 \ : \ x &= \frac{\left(x_1^2 + Dy_1^2\right)}{2} \ \ y = x_1y_1. \\
K = -4 \ : \ x &= \left(x_1^2 + 2\right)\left[\frac{1}{2}\left(x_1^2 + 1\right)\left(x_1^2 + 3\right) - 1\right], \\
y &= \frac{x_1y_1\left(x_1^2 + 1\right)\left(x_1^2 + 3\right)}{2}. \\
K = +4 \ : \ x &= \frac{\left(x_1^2 - 2\right)}{2}, \ \ y = \frac{x_1y_1}{2}, \quad \text{if } x_1 \text{ is even }, \\
x &= x_1\frac{\left(x_1^2 - 3\right)}{2}, \ y = \frac{y_1\left(x_1^2 - 1\right)}{2}, \text{ if } x_1 \text{ is odd.}
\end{aligned}$$

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of Amarthya Kumar Dutta in this volume.

5.2 *Cakravāla* algorithm for the *Varga-prakṛti* $x^2 - Dy^2 = 1$

In his *Bījagaṇita*, Bhāskarācārya gives the *Cakravāla* algorithm for solving the equation $x^2 - Dy^2 = 1$ in four verses (71 -75) [10].

> ह्रस्वज्येष्ठपदक्षेपान् भाज्यप्रक्षेपभाजकान् ।
> कृत्वा कल्प्यो गुणस्तत्र तथा प्रकृतितश्च्युते ॥
> गुणवर्गे प्रकृत्योनेऽथवाल्पं शेषकं यथा ।
> तत्तु क्षेपहृतं क्षेपः व्यस्तः प्रकृतितश्च्युते ॥
> गुणलब्धिः पदं ह्रस्वं ततो ज्येष्ठमथोऽसकृत् ।
> त्यक्त्वा पूर्वपदेक्षेपान् चक्रवालमिदं जगुः ॥
> चतुर्द्व्येकयुतावेवं अभिन्ने भवतः पदे ।
> चतुर्द्विक्षेपमूलाभ्यां रूपक्षेपार्थभावना ॥

> Considering the lesser root, greater root and interpolator (*Kṣepa*) as the dividend, addend and divisor respectively of pulversier) the indeterminate multiplier of it should be so taken as will make the residue of the *prakṛti* diminished by the square of that multiplier or the latter minus the *prakṛti* (as the case may be) to be the least. That residue divided by the (original) interpolator is the (new) interpolator; it should be reversed in sign in case of the subtraction from the *prakṛti*, The quotient corresponding to that value of the multiplier is the (new) lesser root, like wise is obtained the greater root. The same process should be followed putting aside (each time) the previous roots and the interpolator. This process is called *Cakravāla* (or the cyclic method). By this method, there will appear two integral roots corresponding to an equation with $\pm 1, \pm 2$ or ± 4 as interpolator. In order to derive integral roots corresponding to an equation with additive unity from those of the equation with the interpolator ± 2 or ± 4, *Bhāvanā* (should be applied).

We describe algorithm in the following. To solve

$$x^2 - Dy^2 = 1,$$

we have to consider equations of the form

$$x_i^2 - Dy_i^2 = K_i,$$

in the intermediate stages, and also an indeterminate multiplier p_i. Here y_i is the lesser root, x_i is the greater root and K_i is the *kṣepa*. We start with,

$$x_0^2 - Dy_0^2 = K_0, \text{ with } x_0 = 1,\ y_0 = 0,\ K_0 = 1,$$

and set $p_0 = 0$. Given $x_i,\ y_i,\ K_i$ and $p_i,\ y_{i+1}$ and p_{i+1} are obtained by solving the *kuṭṭaka*,

$$y_{i+1} = \frac{y_i\ p_{i+1} + x_i}{|K_i|},$$

with the extra condition that $|p_{i+1}^2 - D|$ is chosen to be minimum. Then x_{i+1} and K_{i+1} can be found from:

$$x_{i+1} = \frac{x_i\ p_{i+1} + Dy_i}{|K_i|},$$

$$K_{i+1} = \frac{p_{i+1}^2 - D}{K_i}.$$

They satisfy:

$$x_{i+1}^2 - Dy_{i+1}^2 = K_{i+1}.$$

Note that the above equation arises by doing *Bhāvanā* between

$$x_i^2 - Dy_i^2 = K_i,$$

$$p_{i+1}^2 - D.1^2 = p_{i+1}^2 - D,$$

and dividing the new $x,\ y$ and K by $|K_i|, |K_i|$ and $|K_i^2|$ respectively.

This process is repeated till we get one of the values $\pm 1, \pm 2, \pm 4$ for the *Kṣepa*. If we obtain the *Kṣepa* to be 1, we have solved the problem. If the *Kṣepa* is -1 or ± 2 or ± 4, *Bhāvanā* can be used to obtain the solution for $x^2 - Dy^2 = 1$, as explained earlier.

Bhāskara II or his commentators have not outlined any proof that *Cakravāla* algorithm always leads to a solution in a finite number of steps. In 1929, A.A.Krishnaswami Ayyangar proved that the *Cakravāla* algorithm always leads to a solution in a finite number of steps [11].

He also showed that the above procedure is equivalent to the one in which the *kuṭṭaka* equation for p_{i+1} is replaced by the condition that K_i divides $p_i + p_{i+1}$, with the other conditions being the same (including minimization of $|p_{i+1}^2 - D|$).

Bhāskara himself considered the example of $D = 61$ in his *Bījagaṇita*. The successive iterated values of p_i, K_i, x_i, y_i are given in Table 1. The smallest solution $x = 1766319049$, $y = 226153980$ is reached after the 14th step, if *Cakravāla* algorithm is applied mechanically. However, just after two steps we find $K = -4$, so that the solution is found immediately using the *Bhāvanā*.

Cakravāla **algorithm for** $x^2 - 61y^2 = 1$

i	p_i	K_i	x_i	y_i
0	0	1	1	0
1	8	3	8	1
2	7	-4	39	5
3	9	-5	164	21
4	6	5	453	5
5	9	4	1523	195
6	7	-3	5639	722
7	8	-1	29718	3805
8	8	-3	469849	60158
9	7	4	2319527	296985
10	9	5	9747957	1248098
11	6	-5	26924344	3447309
12	9	-4	90520989	11590025
13	7	3	335159612	42912791
14	8	1	1766319049	226153980

Table 1

After the first step, we have

$$8^2 - 61 \times 1^2 = 3.$$

In step 2, $8+p_2$ is divisible by 3 and the minimum value of $|p_2^2-61|$ is obtained when $p_2=7$. Then $x_2=39$, $y_2=5$, $K_2=-4$ and we have

$$39^2-61\times 5^2=-4.$$

Since $K=-4$, we can use *Bhāvanā* to obtain

$$\begin{aligned} x &= (39^2+2)\left[\left(\frac{1}{2}\right)\left(39^2+1\right)\left(39^2+3\right)-1\right]=1766319049, \\ y &= \left(\frac{1}{2}\right)(39\times 5)\left(39^2+1\right)\left(39^2+3\right)=226153980. \end{aligned}$$

The Brouncker-Wallis-Euler-Lagrange algorithm for solving the equation $x^2-Dy^2=1$, is based on the continued fraction expansion of $\sqrt{D}$. It can be shown to be identical to *Cakravāla* algorithm, except that the condition

$$\left|D-p_{i+1}^2\right| \text{ is minimum,}$$

is replaced by

$$D-p_{i+1}^2 \text{ is minimum and positive.}$$

The *Cakravāla* algorithm often skips some of the steps encountered in the Brouncker-Wallis-Euler-Lagrange algorithm. Table 2 gives the successive values of p_i, K_i, x_i, y_i in the Brouncker-Wallis-Euler-Lagrange algorithm for $D=61$ where arrows indicate *Cakravāla* steps. The solution is reached after 22 steps using this algorithm, in contrast to 14 steps in the case of *Cakravāla* algorithm. If the number of steps in the *Cakravāla* algorithm and the Brouncker-Wallis algorithm are denoted by n_C and n_B respectively, then it has been shown [12] empirically that for large D,

$$\frac{n_B}{n_C}\approx 1.44.$$

Thus, the *Chakravāla* algorithm is about 44% more efficient for large D. Quadratic indeterminate equations in which ancient Indians did such pioneering work is an active field of research even today and is at the heart of computational mathematics [13].

Brouncker-Wallis-Euler-Lagrange algorithm for $x^2 - 61y^2 = 1$

	I	p_i	K_j	x_i	y_i
→	0	0	1	1	0
	1	7	-12	7	1
→	2	5	3	8	1
→	3	7	-4	39	5
	4	5	9	125	16
→	5	4	-5	164	21
→	6	6	5	453	58
	7	4	-9	1070	137
→	8	5	4	1523	195
	9	7	-3	5639	722
	10	5	12	24079	3083
→	11	7	-	29718	3805
	12	7	12	440131	56353
→	13	5	-3	469849	60158
→	14	7	4	2319527	296985
	15	5	-9	7428430	951113
→	16	4	5	9747967	1248098
→	17	6	-5	26924344	3447309
	18	4	9	63596645	8142716
→	19	5	-4	90520989	11590025
→	20	7	3	335159612	42912791
	21	5	-12	1431159437	183241189
→	22	7	1	1766319049	226153980

Table 2

6 Construction of the Sine-table

In the above figure $AP = R\sin\theta$. This is the Indian *jyā* , where R the radius or *trijyā* is normally chosen such that one minute of arc on the circle corresponds to one unit of distance. Hence R is the number of minutes in a radian, and its value is very close to 3438, which is the value mentioned in the earlier texts. More exact values are mentioned in later texts, especially those of the medieval Kerala school.

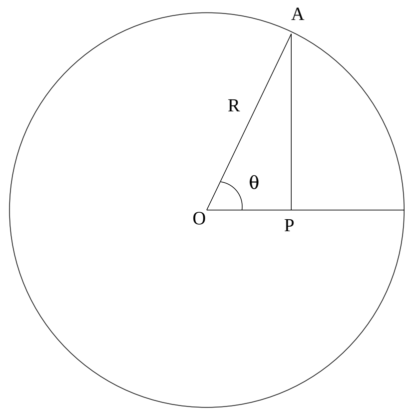

Fig : 5

Normally a quadrant is divided into 24 equal parts, so that each arc bit $\alpha = \frac{90}{24} = 3^0 45' = 225'$. Then the procedure for finding $R \sin i\alpha, \quad i = 1, 2, \ldots 24$ is explicitly given. The R sines of the intermediate angles are to be determined by interpolation.

Āryabhaṭīya gives the explicit algorithm for constructing the sine-table (verse 12, *Gaṇitapāda)*:

> प्रथमाच्चापज्यार्धाद्यैरूनं खण्डितं द्वितीयार्धम् ।
> तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि ॥
>
> The first Rsine divided by itself and then diminished by the quotient gives the second Rsine difference. The same first Rsine diminished by the quotients obtained by dividing each of the preceding R-sines by the first Rsine gives the remaining Rsine-differences.

This tells us that

$$R \sin 2\alpha - R \sin \alpha = R \sin \alpha - \frac{R \sin \alpha}{R \sin \alpha} \qquad (\alpha = 225'),$$

$$R \sin(i+1)\alpha - R \sin i\alpha = R \sin \alpha - \frac{R \sin \alpha + R \sin 2\alpha + \ldots + R \sin i\alpha}{R \sin \alpha}.$$

The second equation is equivalent to the relation:

$$R \sin(i+1)\alpha - R \sin i\alpha = R \sin i\alpha - R \sin(i-1)\alpha - \frac{R \sin i\alpha}{R \sin \alpha}.$$

The values of the 24 *Rsines* are explicitly noted in another verse, and the first *Rsine* is given by

$$R\sin\alpha \approx 225'$$

which obviously uses

$$\sin\alpha \approx \alpha,$$

when α is small. The exact recursion relation for the Rsine differences is:

$$R\sin(i+1)\alpha - R\sin i\alpha = R\sin i\alpha - R\sin(i-1)\alpha - R\sin i\alpha \;\; 2(1-\cos\alpha).$$

Now, $2(1-\cos\alpha) = 0.0042822$, which is approximated in *Āryabhaṭīya* to be

$$\frac{1}{R\sin\alpha} = \frac{1}{225} = 0.00444444.$$

The *Āryabhaṭīya* 's sine values are given in the Table 4. The same procedure with the same values are to be found in many other works including *Sūryasiddhānta* [14]. However the Rsine values given by Govindasvāmi in his commentary on *Mahābhāskarīya* of Bhāskara I which are reproduced in Table 4 are more accurate (correct to five decimal places) [15,16].

Far more accurate values for the sines were given by Mādhava of Saṅgamagrāma (1340-1425), the father figure of the Kerala school of astronomy and mathematics. These are cited in the two commentaries on Nīlakaṇṭha Somayāji's *Tantrasaṅgraha* (c.1500)[17,18] namely *Laghuvivṛtti* [17] and *Yuktidīpikā* [18], both composed by Śaṅkara Vāriyar.

These results are based on the series expansion for $\sin\theta$, which we write in the following form:

$$R\sin\theta = R\theta - \frac{(R\theta)^3}{3!R^2} + \frac{(R\theta)^5}{5!R^4} - \frac{(R\theta)^7}{7!R^6} + \frac{(R\theta)^9}{9!R^8} - \frac{(R\theta)^{11}}{11!R^{10}} + \ldots,$$

where $R\theta$ is the arc in minutes [19]. Explicit values of the magnitudes of the terms starting in the reverse from sixth and up to the second in (the RHS of) the above equation, given by Mādhava when the arc $R\theta = 5400' = 90°$, are mentioned in *Yuktidīpikā* in the *kaṭapayādi* notation. The verse giving these values is quoted below[18].

विद्वांस्तुन्नबलः कवीशनिचयः सर्वार्थशीलस्थिरः
निर्विद्धाङ्गनरेन्द्ररुङ्निगदितेष्वेषु क्रमात् पञ्चसु ।

These values along with the modern values for comparison are listed in Table 3. The last column in this table was computed using $R = 3437.747$ and $R\theta = 5400$.

Term no. in RHS	Sanskrit equivalent in *kaṭapayādi*	Mādhava's value according to *Yuktidīpikā*	Modern value
VI	विद्वान्	$0'0''44'''$	$0'0''44.54'''$
V	तुन्नबलः	$0'33''6'''$	$0'33''5.6'''$
IV	कवीशनिचयः	$16'05''41'''$	$16'05''40.87'''$
III	सर्वार्थशीलस्थिरः	$273'57''47'''$	$273'57''47.11'''$
II	निर्विद्धाङ्गनरेन्द्ररुक्	$2220'39''40'''$	$2220'39''40.10'''$

Table 3

We find that the values given by Mādhava are indeed very accurate. For an arbitrary arc $R\theta$ (in minutes), the procedure is given in the later half of the verse quoted above [18]:

आधस्त्यात् गुणितादभीष्टधनुषः कृत्या विहृत्यान्तिमस्-
याप्तं शोध्यमुपर्युपर्यथ धनेनैवं धनुष्यन्ततः ॥

It is the following formula for $R\sin\theta$ that is delineated by the above verse:

$$R\sin\theta = R\theta - \beta^3(2220'39''40''') + \beta^5(273'57''47''') - \beta^7(16'5''41''') + \beta^9(0'33''6''') - \beta^{11}(0'0''44'''),$$

where $\beta = \frac{R\theta}{5400}$. The 24 Rsines corresponding to $R\theta = 225', 450', 675' \ldots$ were also explicitly stated by Mādhava in the *kaṭapayādi* notation and they have been given in *Laghuvivṛtti* in the set of verses beginning with **श्रेष्ठं नाम वरिष्ठानां हिमाद्रिर्वेदभावनः** and ending with **तत्परादिकलान्तास्तु महाज्या माधवोदिताः ।** They coincide with the modern values up to "thirds" (corresponding to an accuracy of sines up to seventh or eighth decimal places).

	R sin θ according to		
θ in min.	*Āryabhaṭīya*	Govindasvāmi	Mādhava(also Modern)
225	225	224 50 23	224 50 22
450	449	448 42 53	448 42 58
675	671	670 40 11	670 40 16
900	890	889 45 08	889 45 15
1125	1105	1105 01 30	1105 01 39
1350	1315	1315 33 56	1315 34 7
1575	1520	1520 28 22	1520 28 35
1800	1719	1718 52 10	1718 52 24
2025	1910	1909 54 19	1909 54 35
2250	2093	2092 45 46	2092 46 03
2475	2267	2266 38 44	2266 39 50
2700	2431	2430 50 54	2430 51 15
2925	2585	2584 37 43	2584 38 06
3150	2728	2727 20 29	2727 20 52
3375	2859	2858 22 31	2858 22 55
3600	2978	2977 10 09	2977 10 34
3825	3084	3083 12 51	3083 13 17
4050	3177	3175 03 23	3176 03 50
4275	3256	3255 17 54	3255 18 22
4500	3321	3320 36 02	3320 36 30
4725	3372	3371 41 01	3371 41 29
4950	3409	3408 19 42	3408 20 11
5175	3431	3430 22 42	3430 23 11
5400	3438	3437 44 19	3437 44 48

Table 4

In Table 4, we compare the values of the *Rsines* in *Āryabhaṭīya* , Govindasvāmin's commentary of *Mahābhāskarīya*, and Mādhava's values as stated in *Laghuvivṛtti* [16] and the modern values.

It may be noted that the table gives the Rsine values corresponding to arc lengths which are multiples of 225'. In the third and the fourth columns, the values are in minutes, seconds and thirds. Note that the modern values coincide with the Mādhava's values in the last column.

7 Accurate computation of π using an error-minimization algorithm

It is well known that an infinite series for π was first given by the Kerala mathematicians, who invariably ascribe the result to Mādhava of Saṅgamagrāma (1340 - 1425 AD):

$$\frac{Circumference}{Diameter} = \pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots \qquad).$$

An ingenious geometrical proof of this is to be found in the celebrated Malayalam text *Yuktibhāṣā* of Jyeṣṭhadeva (c.1530) [20,21]. As is well known, this series converges very slowly. An algorithm for accurate and efficient computation of π using the technique of correction terms was also given by Mādhava. This result is cited in *Yuktidīpikā* of Śaṅkara Vāriyar [22].

व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते ।
त्रिशरादि-विषमसङ्ख्याभक्तमृणं स्वं पृथक् क्रमात् कुर्यात् ॥
यत्सङ्ख्यात्र हरणे कृते निवृत्ता हृतिस्तु जामितया ।
तस्या ऊर्ध्वगता या समसङ्ख्या तद्दलं गुणोऽन्ते स्यात् ॥
तद्वर्गो रूपयुतो हारो व्यासाब्धिघाततः प्राग्वत् ।
ताभ्यामाप्तं स्वमृणे कृते घने क्षेप एव करणीयः ॥
लब्धः परिधिः सूक्ष्मः बहुकृत्वो हरणतोऽतिसूक्ष्मः स्यात् ॥

> The diameter multiplied by four and divided by unity (is found and stored). Again the products of the diameter and four are divided by the odd numbers like three, five, etc., and

> the results are subtracted and added in order (to the earlier stored result).
>
> Take half of the succeeding even number as the multiplier (*guṇa* term) at whichever (odd) number the division process is stopped, because of boredom (by the slow converging process). The square of that (even number) added to unity is the divisor. The ratio has to be multiplied by the product of the diameter and four as (stated) earlier.
>
> The result obtained has to be added if the earlier term (in the series) has been subtracted and subtracted if the earlier term has been added. The resulting circumference is very accurate; in fact more accurate than the one which may be obtained by continuing the division process (with a large number of terms in the series).

Essentially,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \frac{1}{p} \mp \frac{\left(\frac{p+1}{2}\right)}{(p+1)^2 + 1},$$

to a fairly good approximation.

Describing different ways by which better approximations can be obtained, finally Śaṅkara Vāriyar states a more accurate correction term [22]:

> एभ्यः सूक्ष्मतरोऽन्यो विलिख्यते कश्चनापि संस्कारः ।
> अन्ते समसङ्ख्यादलवर्गः सैको गुणः स एव पुनः ॥
> युगगुणितो रूपयुतः समसङ्ख्यादलहतो भवेद् हारः ।
> त्रिशरादि विषमसङ्ख्याहरणात् परमेतदेव वा कार्यम् ॥

> Now a correction term better than the others will be presented. Here square of half of the even term increased by unity, is the multiplier. The same (square) increased by unity and multiplied by four, and further multiplied by half the even term is the divisor. This correction term should be applied after dividing by the odd numbers three, five, etc.

In other words, a much better approximation for $\frac{\pi}{4}$ is:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \frac{1}{p} \mp \frac{\left(\frac{p+1}{2}\right)^2 + 1}{[(p+1)^2 + 4 + 1]\left(\frac{p+1}{2}\right)}.$$

In *Yuktibhāṣā* , these correction terms are derived using an error-minimization algorithm [21,23]. We give a summary of this procedure as explained in the above references. If the series is truncated at odd number $p-2$ with a correction term $\frac{l}{S_1}$, $\frac{\pi}{4}$ can be written as:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots - \frac{1}{p-2} + \frac{1}{S_1}.$$

On the other hand, if the series is truncated at p with a correction term $\frac{l}{S_2}$, we will have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots - \frac{1}{p-2} + \frac{1}{p} - \frac{1}{S_2}.$$

If the correction terms are exact, then both should yield the same result. That is,

$$\frac{1}{S_1} = \frac{1}{p} - \frac{1}{S_2} \qquad \text{or} \qquad \frac{1}{S_1} + \frac{1}{S_2} = \frac{1}{p}.$$

$S_1 = S_2 = 2p$ actually satisfies the equality. However both the corrections should follow the same rule. If $S_1 = 2p-2$ and $S_2 = 2p+2$, that is S is twice the even number above the last odd number in the series, then the *Sthaulya* or inaccuracy is

$$\Delta = \frac{1}{S_1} + \frac{1}{S_2} - \frac{1}{p} = \frac{1}{2p-2} + \frac{1}{2p+2} - \frac{1}{p} = \frac{4}{4p^3 - 4p},$$

and it can be shown that the error (corresponding to this choice of S) is minimum. For other choices such as $S_1 = 2p-3$ and $S_2 = 2p+1$, the inaccuracy Δ will have a term proportional to p in the numerator, whereas the denominator $\approx 4p^3$. So the inaccuracy Δ will be much larger than its value for $S_1 = 2p-2$, when p is large. By choosing $S_2(p)$ of the form,

$$S_2(p) = \frac{1}{2p + 2 + \frac{A}{2p+2}},$$

it can be shown that the inaccuracy is reduced further and inaccuracy Δ is minimum at $A = 4$. So,

$$S_2(p) = \frac{1}{(2p+2) + \dfrac{4}{2p+2}} = \frac{\left(\dfrac{p+1}{2}\right)}{(p+1)^2+1}.$$

This leads us to

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots \pm \frac{1}{p} \mp \frac{\left(\dfrac{p+1}{2}\right)}{(p+1)^2+1},$$

which is the first rule given in *Yuktidīpikā* . Proceeding in the same manner, the error is further reduced and will be minimum when we take

$$S_2(p) = \frac{1}{(2p+2) + \dfrac{4}{2p+2+\dfrac{16}{2p+2}}},$$

which leads to

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots \pm \frac{1}{p} \mp \frac{\left(\dfrac{p+1}{2}\right)^2 + 1}{[(p+1)^2+4+1]\left(\dfrac{p+1}{2}\right)}.$$

This is the more accurate second rule given in *Yuktidīpikā* . When $p = 99$ (that is, considering 50 terms in the series), the resulting value of π is found to be accurate to 11 decimal places. This is same as the value $\pi \approx 3.14159265359222\ldots$ given by the rule, विबुधनेत्रगजाहिहुताशन. . . , attributed to Mādhava by Nīlakantha in his *Āryabhatīyabhāṣya Gaṇitapāda*, verse 2.

When the error-minimization algorithm is continued along the above lines two steps further,

$$S_2(p) = \frac{1}{(2p+2) + \dfrac{4}{2p+2+\dfrac{16}{2p+2+\dfrac{32}{2p+2+\dfrac{64}{2p+2}}}}}.$$

When $p = 99$, this yields $\pi = 3.141592653589793241$, which is correct to 17 decimal places. This is the same as the value for π stated by Rājā Śaṅkara Varma in his *Sadratnamālā* (c.1823).

Acknowledgement

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Algorithms in Indian Astronomy

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Abstract

Indian Astronomy is rich in algorithms. The algorithms presented in the Indian astronomical texts have varying degrees of complexities starting from the simple *trairāśika* rule, to the treatment of parallax in a solar eclipse or the computation of the elevation of lunar cusps. In the present article we will discuss a few algorithms that are representative of the ingenuity and continuity of the Indian astronomical tradition. We start with the interpolation formula presented by Brahmagupta (c.665 AD) and then proceed to describe a select few algorithms from *Tantrasaṅgraha* of Nīlakaṇṭha composed in 1500 AD. Here we present the algorithm for the calculation of time from shadow measurements and the exact algorithm for the computation of *lagna* and the time for the duration of an eclipse. We also comment on the iterative process known as *aviśeṣakarma* which aims at circumventing the problem of interdependencies among several variables.

1 Introduction

It is not uncommon to find words which originate with a different connotation and in due course pick up a completely different connotation. The word *algorithm* forms a good example of this. Its origin can be traced back to the Persian mathematician, al-Khwārazmī (800-847 AD). It is quite interesting to note the observations made by D.E. Knuth in this context [1]:

> In the middle ages, abacists computed on the abacus and algorists computed by algorism. Following the middle ages, the origin of this word was in doubt, and early linguists attempted to guess at its derivation by making combinations like algiros [painful] + arithmos [number]; others said no, the word comes from "King Algor of Castile". Finally, historians of mathematics found the true origin of the word algorism: it comes from the name of a famous Persian textbook author, Abu Ja'far Mohammed ibn Musa al-Khwārizmī ($\sim$ 825 AD) - literally, "Father of Jafar, Mohammed, son of Moses, native of Khwārizm." Khwārizm is today the small Soviet city of Khiva. al-Khwārizmī wrote the celebrated book Kitab al jabr w'al-muqabala ("Rules of restoration and reduction"); another word, "algebra", stems from the title of his book, although the book wasn't really very algebraic.

Gradually the form and meaning of "algorism" got distorted; The change from "algorism" to "algorithm" is any body's guess. The remarks by the well-known historian C.B.Boyer in this context are also noteworthy [2].

> ...... when subsequently Latin translations of his (al Khwārizmī's) work appeared in Europe, careless readers began to attribute not only the book but also the numeration to the author. The new notation came to be known as that of al-Khwārizmī, or more carelessly, algorismi; ultimately the scheme of numeration making use of Hindu numerals came to be called simply algorism or algorithm.

The terms process, method, technique, procedure, routine, and so on all essentially refer to a sequence of operations to be carried out to accomplish a given task. The word "algorithm" though similar, connotes something more. For a procedure to be termed algorithm it must terminate after n steps and upon termination it must yield a sensible result. However, this is not true of all procedures. In this sense, *all algorithms are procedures*; but *all procedures are not algorithms.*

Indian astronomy is essentially algorithmic in nature. The algorithms presented are precise and fairly sophisticated [3]. Some of them are amazingly accurate [4]. We shall illustrate these points by considering a few examples.

2 Brahmagupta's interpolation formula

Interpolation is the art of reading between the lines in a table. The rule of *trairāśika* [5,6] employed in Indian astronomy is close to what is known as the first-order interpolation in modern parlance. This technique has been extensively applied to solve a variety of mathematical and astronomical problems, beginning from the evaluation of sine function to the calculation of eclipses, at least from the time of *Āryabhaṭīya* (c.499 AD).

It is quite interesting to note that Brahmagupta introduced the second order interpolation formula to determine more precise value of the sine function, called *jyā* , for an arbitrary angle, from the set of tabulated values of sine given at fixed intervals. The following verse from his famous work *Khaṇḍakhādyaka* (c.665 AD) explains the algorithm [7]:

> गतभोग्यखण्डकान्तरदलविकलघातशतैर्नवभिराप्तया ।
> तद्युतिदलं युतोनं भोग्यादूनाधिकं भोग्यम् ॥

> Multiply the residual arc left after division by 900′ by half the difference of the tabular difference passed over and that to be passed over and divide by 900′; by the result increase or decrease, as the case may be, half the sum of the same two tabular differences; the result which, whether less or greater than the tabular difference to be passed, is the true tabular difference to be passed over.

Generally in Indian astronomical texts, the interval chosen for tabulating values of sines is $225'$. However, Brahmagupta has chosen the interval to be $15^\circ = 900'$ for the sake of simplicity, as *Khaṇḍakhādyaka* is a *karaṇa* text.[1] This explains the number 900 figuring in the above verse.

[1] Of the three kinds of texts, *Siddhānta, Tantra and Karaṇa*, the last is meant to serve as a simple manual used for quick computations.

Before we represent the content of the above verse in a mathematical form, it would be useful to introduce the terminologies and the notation employed. Let the equal intervals chosen be represented by α. That is,

$$x_n - x_{n-1} = x_{n+1} - x_n = \alpha = 900, \tag{1}$$

where the variable x denotes the angle and $x_i^{\prime}s$ are multiples of $900'$. The term *Bhogya* refers to the present interval between x_n and x_{n+1}. The term *khaṇḍaka* refers to the I order tabular differences in the sine values and we denote it by Δ_n.

$$\Delta_n = f(x_n) - f(x_{n-1}). \tag{2}$$

Let us suppose that the value of the function at three points $f(x_{n-1})$, $f(x_n)$, and $f(x_{n+1})$ are known, and it is required to find out $f(x_n + \beta\alpha)$. Now, the formula given by Brahmagupta may be written as

$$f(x_n + \beta\alpha) = f(x_n) + \frac{\beta\alpha}{\alpha}\left[\frac{(\Delta_n + \Delta_{n+1})}{2} - \beta\frac{(\Delta_n - \Delta_{n+1})}{2}\right], \tag{3}$$

where $0 < \beta < 1$. With a little algebraic manipulation, and suitably rearranging the terms, the above equation may be rewritten as

$$f(x_n+\beta\alpha) = f(x_n)+\frac{\beta\alpha}{\alpha}\times\Delta_{n+1}+\frac{\beta\alpha}{\alpha}\left(\frac{\beta\alpha}{\alpha}-1\right)\times\frac{(\Delta_n-\Delta_{n+1})}{2}. \tag{4}$$

Clearly, the formula given by Brahmagupta is identical with the standard quadratic interpolation formula [8].

3 Algorithm for finding the time from shadow

The technique of making a fairly good estimate of the time from the shadow of an object cast by the Sun has been in vogue from time immemorial. Different cultures and traditions across the world have devised simple instruments for this purpose. In the Indian astronomical tradition the instrument used is called *śaṅku* .

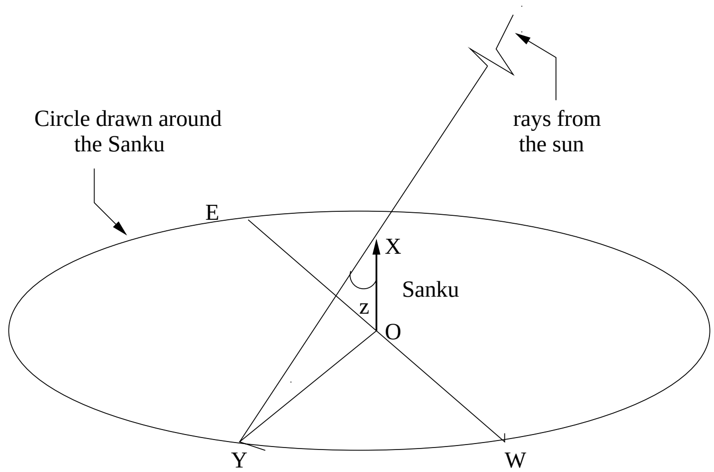

Figure 1: Zenith distance and the length of the shadow.

The *śaṅku* essentially consists of a rod of suitable thickness and height. Generally the height is taken to be 12 *aṅgulas* . For performing experiments with *śaṅku* , it must be placed at the centre of a circle as shown in Fig.1. Here, OX represents the *śaṅku* and OY is its shadow cast by the sun. If the longitude of the sun, λ is known at a given instant, then its declination, δ at that instant can be calculated using the relation (see Fig.2)

$$\sin\delta = \sin\epsilon \sin\lambda, \tag{5}$$

where ϵ is the obliquity of the ecliptic. The procedure for obtaining accurate values of the observer's latitude, ϕ, are discussed in detail in several Indian astronomical texts [9].

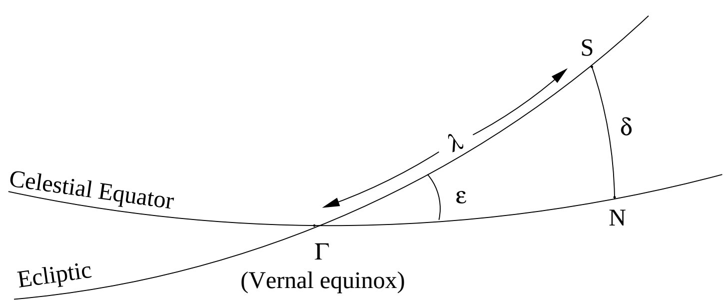

Figure 2: Declination of the Sun.

The formula given by Nīlakaṇṭha in his celebrated work *Tantrasaṅgraha* (c.1500 AD) for determining the time from shadow measurement turns out to be essentially a function of ϕ, δ and z, where z is the zenith distance of the sun. Since ϕ and δ are already found through shadow measurements, only z needs to be known. For this, consider the triangle OXY in Fig.1. It can be easily seen that

$$\begin{aligned} \sin z &= \frac{OY}{XY} \\ \text{or} \qquad z &= \sin^{-1}\left(\frac{OY}{XY}\right), \end{aligned} \tag{6}$$

where XY is the hypotenuse given by $\sqrt{12^2 + OY^2}$. Thus, the zenith distance of the sun can also be obtained at any time by measuring the shadow cast by the *śaṅku* .

The algorithm for finding the time from shadow measurements is presented by Nīlakaṇṭha as follows [10]:

> व्यासार्धघ्नात् ततः शङ्कोः लम्बकाप्तं त्रिजीवया ॥
> हत्वा द्युज्याविभक्ते तत् चरज्या स्वर्णमेव च ।
> याम्योदग्गोलयोस्तस्य चापे व्यस्तं चरासवः ॥
> संस्कार्या गतगम्यास्ते पूर्वापरकपालयोः ।
>
> The *śaṅku* is multiplied by radius (*trijyā*) and divided by *lambaka* . This is further multiplied by *trijyā* and divided by *dyujyā* . To this quantity the *carajyā* is applied positively or negatively depending upon whether the Sun is in the southern or the northern hemisphere.
>
> To the arc of the result, the *carāsavaḥ* (ascensional difference) has to be applied in the reverse order. This gives the time that has elapsed or yet to elapse in the eastern and the western half of the hemisphere.

Before we express the above verse in mathematical notation we introduce a few technical terms appearing in the above verse. The term *trijyā* refers to R,[2] *lambaka* is $R\cos\phi$ and *dyujyā* is $R\cos\delta$. The word *śaṅku*

[2] It is usually taken to be the measure of one radian in minutes. That is $R \approx 3438'$.

in the first line of the quotation is used to denote $R\cos z$.[3] *Carajyā* is $R\sin\Delta\alpha$. With this, the the first two lines of the verse translate to the relation

$$R\sin\theta = \left[\frac{R\cos z}{\cos\phi\cos\delta} \pm R\sin\Delta\alpha\right] \tag{7}$$

In the latter half of the quotation given above it is mentioned that to the arc of the above (θ), the ascensional difference, which is the angular separation between the vernal equinox and the celestial object measured along the celestial equator (see Fig.3), has to be applied in the reverse order to obtain the required time t. That is,

$$t = \theta \mp \Delta\alpha. \tag{8}$$

Substituting for θ we have

$$t = (R\sin)^{-1}\left[\frac{R\cos z}{\cos\phi\cos\delta} \pm R\sin\Delta\alpha\right] \mp \Delta\alpha. \tag{9}$$

The above result can be easily understood using the tools of spherical trigonometry.[4] For this, consider Fig.3 where S is the Sun on its diurnal path whose zenith distance is z, corresponding to the arc ZS. The point where the sun sets in the western part of the horizon is denoted by S_t. The segment PW is a part of the 6 $0'$ clock circle, and the angle $Z\hat{P}W = 90^o$. Applying four-part formula to the triangle PWS_t, it can be shown that, the ascensional difference $\Delta\alpha$ is given by the relation (well known in Indian Astronomy)

$$\sin\Delta\alpha = \tan\phi\tan\delta. \tag{10}$$

Further, it may be noted that $S\hat{P}S_t = \theta + \Delta\alpha$ is the angle to be covered by the Sun from the given instant up to the sunset. This angle divided by 6 and 15 give the time that is to elapse before sunset in hours and *nāḍikās* (time unit of approximately 24 minutes) respectively.

---

[3]Generally $R\cos z$, which is the perpedicular distance of the sun from the horizon, is called *mahāśaṅku* in order to distinguish it from the gnomon (*dvādaśāṅgulaśaṅku*) used for shadow measurement. But sometimes *mahāśaṅku* would be simply referred as *śaṅku* , as in the above verse, and what it refers to would be clear from the context.

[4]Detailed demonstrations of the algorithms enunciated in Indian astronomical and mathematical texts are given in the famous Malayalam work *Gaṇita-Yuktibhāṣā* (c.1530 AD) of Jyeṣṭhadeva [Ref.6, Vol.I & II].

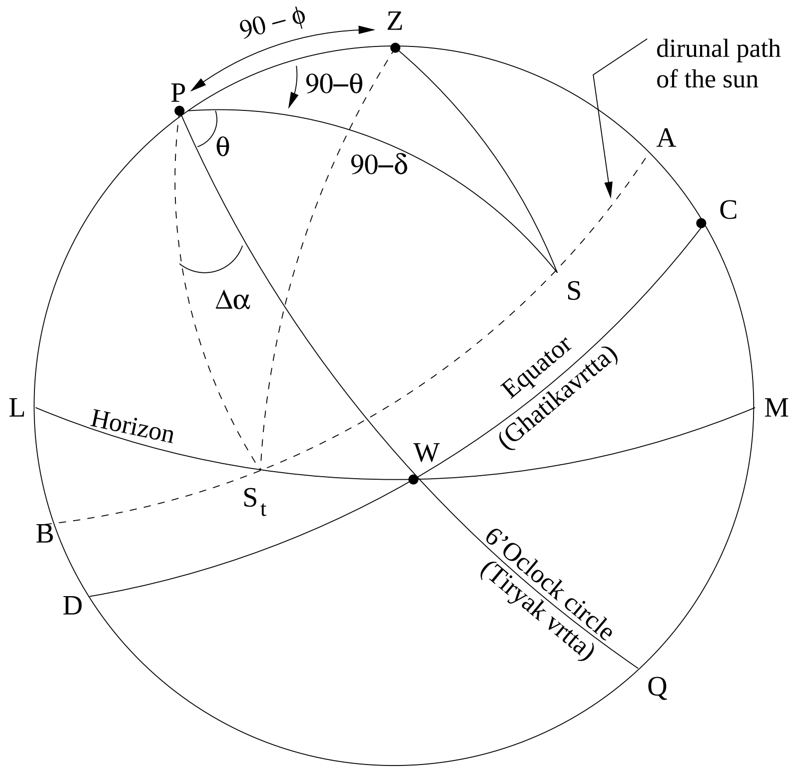

Figure 3: Determination of time from shadow measurements.

Applying the cosine formula to the spherical triangle PZS, we have

$$\cos z = \sin\phi \sin\delta + \cos\phi \cos\delta \sin\theta,$$

or,

$$\begin{aligned} \sin\theta &= \frac{\cos z}{\cos\phi\cos\delta} - \tan\phi\tan\delta \\ &= \frac{\cos z}{\cos\phi\cos\delta} - \sin\Delta\alpha. \end{aligned}$$

Hence, the time, t that is yet to elapse before sunset in angular measure is

$$t = \theta + \Delta\alpha = \sin^{-1}\left[\frac{cosz}{\cos\phi\cos\delta} - \sin\Delta\alpha\right] + \Delta\alpha, \qquad (11)$$

which is the same as Eq.(9) given by Nīlakaṇṭha .

The method to determine the time elapsed after sunrise is exactly similar. It is worth mentioning in this context that, in finding z, the correction due to the finite size of the Sun and its parallax have also been taken into account by Nīlakaṇṭha [11]. If z' were the apparent zenith distance, then the true zenith distance z is given by

$$z = z' + \Delta\theta \qquad with \qquad \Delta\theta = d_s - p,$$

where d_s is the angular semi-diameter of the Sun and p its parallax. As $\Delta\theta$ is small,

$$\begin{aligned} R\cos z &= R\cos(z' + \Delta\theta) \\ &\approx R\cos z' - \Delta\theta\left(\frac{R\sin z'}{R}\right). \end{aligned} \tag{12}$$

This is precisely what is stated to be the *mahāśaṅku*, $R\cos z$, in an earlier verse [12], which is to be used in the expression (Eq.9) for determining the time from the shadow.

4 Algorithm for finding *lagna*

Lagna refers to the orient ecliptic point, that is, the point of the ecliptic which intersects with the eastern horizon at any desired instant. Nīlakaṇṭha while discussing the procedure for finding *lagna*, in his *Tantrasaṅgraha* first presents the conventional method, which could be found in many of the earlier texts on Indian astronomy. Later, pointing out that this procedure would give only approximate results, he proceeds to give an exact algorithm for finding the *lagna* .

4.1 Conventional method

The following verses of *Tantrasaṅgraha* describe the standard procedure used for the computation of *lagna* at any desired instant [13].

संस्कृतायनभानूत्थराशिगन्तव्यलिप्तिकाः ॥
तद्राशिस्वोदयप्राणहता राशिकलाहृताः ।
असवो राशिशेषस्य गतासुभ्यस्त्यजेच्च तान् ॥

उत्तरोत्तरराशीनां प्राणाः शोध्याश्च शेषतः ।
पूरयित्वा रवौ राशिं क्षिपेद्राशींश्च तावतः ॥
विशुद्धा यावतां प्राणाः, शेषास्त्रिंशद्गुणात् पुनः ।
तदूर्ध्वराशिमानाप्तान् भागान् क्षिप्त्वा रवौ तथा ॥
षष्टिघ्नाच्च पुनः शेषात् तन्मानाप्तकला अपि ।
एवं प्राग्लग्नमानेयं अस्तलग्नं तु षड्भयुक् ॥
व्यत्ययेनायनं कार्यं मेषादित्वप्रसिद्धये ।

From the longitude of the Sun corrected for *ayana*, the number of minutes to be elapsed in that *rāśi* (are calculated). This is multiplied by the duration of rising of that *rāśi* and is divided by the number of minutes in a *rāśi* . This gives that *prāṇas* for the remaining *rāśi* to rise and that has to be subtracted from the duration elapsed (since the sunrise). From the remainder (h'), the durations of rising of the *rāśis* that follow have to be subtracted. Having added the degrees remaining in that *rāśi* to the Sun, the other *rāśis* (degrees corresponding to *rāśis*), as many number of them, whose rising times were subtracted are also added. The remaining *prāṇas* (r) are multiplied by 30 and divided by the duration of rising of that *rāśi* . The result obtained is once again added to the Sun.
The remainder when multiplied by 60 gives the result in minutes. Thus, *prāglagna* , the orient ecliptic point should be obtained. The *astalagna* , setting ecliptic point is obtained by adding six signs to that.
To know the longitude of the ecliptic points from the *meṣadi*, the *ayana* correction has to be applied reversely.

The procedure described here may be understood with the help of Fig.4. Here S represents the Sun in the eastern part of the hemisphere, Γ the vernal equinox. R_1, R_2 etc., are the ending points of the first *rāśi* (sign), second *rāśi* and so on. h refers to the time elapsed after the sunrise and H the hour angle of the the Sun.

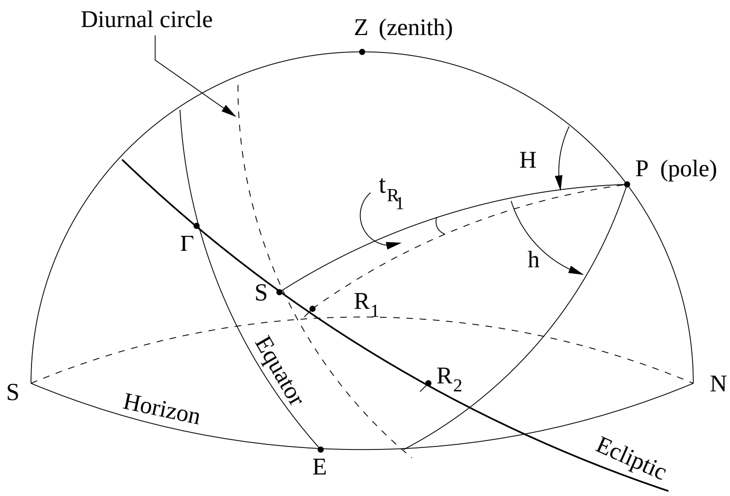

Figure 4: Determination of *prāglagna* (orient ecliptic point) – by the conventional method

Let λ_s be the *sāyana* longitude[5] of the Sun. Suppose the Sun is in the i^{th} *rāśi* (in the Fig.4, it is shown to be in the first *rāśi*), whose rising time at the observer's location is given by T_i. If θ_{R_i} be the angle remaining to be covered by the Sun in that *rāśi* (in minutes), then the time required for that segment of the *rāśi* to come above the horizon is given by

$$t_{R_i} = \frac{\theta_{R_i} \times T_i}{30 \times 60}, \tag{13}$$

where T_i is in *ghaṭikas* . A factor of 60 in the denominator indicates that the result t_{R_i} is expressed in *prāṇas*.[6] Subtracting this time t_{R_i} from the time elapsed since sunrise h, we have

$$h' = h - t_{R_i}.$$

From h' the time required for the subsequent *rāśis* to rise, T_{i+1}, T_{i+2}, etc., are subtracted till the remainder r remains positive. That is,

$$r = h' - T_{i+1} - T_{i+2} \ldots - T_{i+j-1} \qquad (r\ +ve)$$

[5]This refers to the longitude measured from the vernal equinox. Here, it may be noted that in Indian astronomy, *nirayaṇa* longitudes are more commonly used. They refer to the longitude measured from a fixed point, which is generally taken to be the beginning point of star called *Aśvini*. It is also referred to as *meṣādi* as in the last line of the Sanskrit quotation given above.

[6]*Ghaṭikā* is a measure which is close to 24 minutes and *prāṇa* $\approx$ 4 seconds.

Suppose we are in the $i+j^{th}$ *rāśi* whose rising time is T_{i+j}. Then, the portion of R_{i+j}, which would have come above the horizon in the remaining time r is given by

$$\theta_{E_{i+j}} = \frac{r \times 30}{T_{i+j}} \qquad (in\ deg.). \tag{14}$$

Now, the longitude of the *prāglagna* (L) is given by

$$L = \lambda_s + \theta_{R_i} + 30 + 30 + \ldots + \theta_{E_{i+j}}. \tag{15}$$

If so, then the *astalagna* is given by

$$astalagna\ = pr\bar{a}glagna\ + 180^\circ, \tag{16}$$

because the horizon divides the ecliptic exactly into two parts. The *lagnas* obtained by the above procedure are *sāyana lagnas*. To obtain the *nirayaṇa* ones, one needs to subtract the *ayanāṃśa*.

After describing the above method, Nīlakaṇṭha remarks that it is only approximate [14]:

> एकस्मिन्नपि राशौ तु क्रमात् कालो हि भिद्यते ॥
> तेन त्रैराशिकं नात्र कर्तुं युक्तं यतस्ततः ।
> एवमानीतलग्नस्य स्थूलतैव न सूक्ष्मता ॥
>
> The (rising) time gradually differs even in the same *rāśi* . Hence, applying rule of three here, is not appropriate. Therefore the *lagna* obtained by the above procedure will only be approximate and not exact.

4.2 Exact algorithm

Subsequently, Nīlakaṇṭha gives an exact algorithm for finding *lagna*. This is presented in several steps. First he finds the *kālalagna* (defined in the following verse [15]), which is nothing but the time difference between the desired instant and the time of rising of the vernal equinox Γ.

सायनर्कभुजाप्राणाः प्राग्वत् स्वचरसंस्कृताः ।
```द्युगतप्राणसंयुक्तः कालो ``` ।

> The right ascension of the *sāyana* Sun corrected by the ascensional difference ... this added to the *prāṇas* elapsed gives the *kālalagna* .

The time difference between the sunrise and the rising of the vernal equinox Γ is $\alpha - \Delta\alpha$ (see Fig.5), where α is the right ascension of the Sun and $\Delta\alpha$ the ascensional difference. When this is added to the time elapsed after sunrise, we obtain the *kālalagna* at the desired instant, which we denote by L'.

$$\textit{kālalagna} \ = L' = \text{Time elapsed after sun rise} + (\alpha - \Delta\alpha). \qquad (17)$$

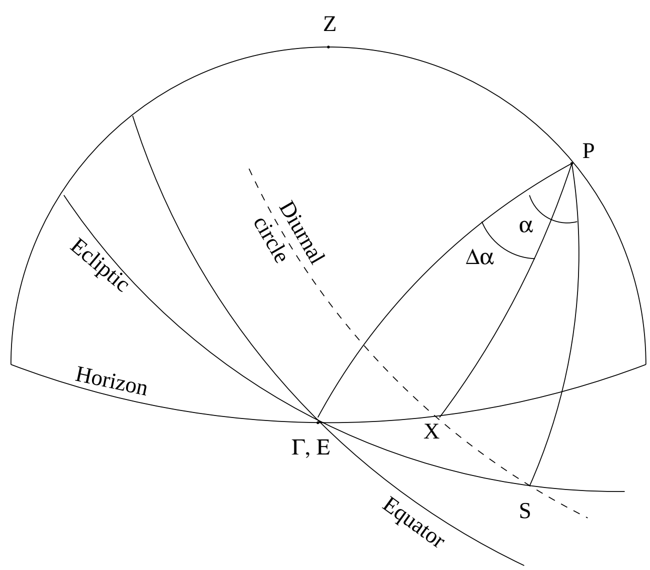

Figure 5: Determination of *kālalagna* .

Now, *vitribhalagna* is the point on the ecliptic which is 90° away from the *lagna*. Consider Fig.6, where S is the Sun, Γ is the vernal equinox and L is the orient ecliptic point, whose longitude is the *lagna*. V is the *vitribhalagna* and K is the pole of the ecliptic. *Dṛkṣepa* is sine of
```

the zenith distance of the *vitribhalagna* ($R\sin ZV$). When Γ is on the horizon, K is on the meridian. It can be easily seen that the *kālalagna* is the hour angle of K, or $L' = Z\hat{P}K$. Now,

$$ZK = KV - ZV = 90 - ZV,$$

as K is the pole of the ecliptic. Also, $KP = \epsilon$ and $ZP = 90 - \phi$. Then, using the cosine formula we obtain the following expression for *dṛkṣepa*

$$R\sin ZV = R\cos ZK = R\cos\epsilon\sin\phi + R\cos\phi\sin\epsilon\cos L'. \qquad (18)$$

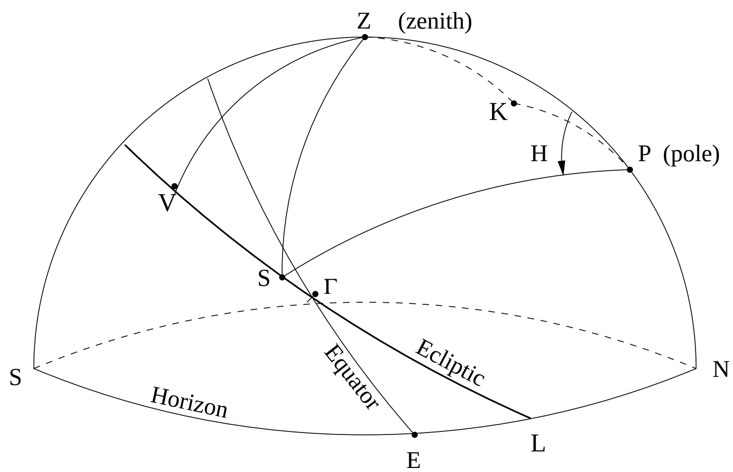

Figure 6: Determination of *prāglagna* (orient ecliptic point) - exact method.

This is essentially what is stated in the following verse [16].

> अन्त्यद्युज्याहताक्षाद्यत् त्रिज्याप्तं यश्च लम्बकः ॥
> काललग्नोत्थकोटिघ्नः करार्थाब्ध्युरगैर्हृतः ।
> दृक्क्षेपस्तद्विदैक्यं च काले कर्किमृगादिके ॥

The *akṣa* multiplied by *antyadyujyā* and divided by *trijyā*, and, *lambaka* multiplied by the *koṭi* of the *kālalagna* and divided by 8452 (are kept separately). The *dṛkṣepa* is the difference or sum of the two depending upon whether *kālalagna*

> is within the 6 signs beginning from *karkaṭaka* or *mṛga* Capricorn.

Here Nīlakaṇṭha defines two intermediate quantities x and y whose sum or difference gives the expression for the sine of the zenith distance of the the *vitribhalagna* . They are given by

$$x = \frac{antyadyujy\bar{a} \times ak\d{s}a}{trijy\bar{a}} = \frac{R\cos\epsilon \times R\sin\phi}{R},$$

and,

$$y = \frac{lambaka \ \times ko\d{t}i \text{ of } k\bar{a}lalagna}{8452} = \frac{R\cos\phi \times R\cos L' \times \sin\epsilon}{R}.$$

It may be noted in the above equation that $\frac{\sin\epsilon}{R} = \frac{\sin 24}{3438}$ is taken to be $\frac{1}{8452}$. Now, *dṛkṣepa* ($= R\sin ZV$, refer Fig.6) is given to be

$$d\d{r}k\d{s}epa \quad = \quad x \pm y$$

Substituting for x and y we have,

$$R\sin ZV = R\cos\epsilon\sin\phi \pm R\cos\phi\cos L'\sin\epsilon, \tag{19}$$

which is same as Eq.(18). From this, $\cos ZV$ is to be calculated. With this, the *lagna* is to be found as follows [17]:

> मध्याह्नाद्वा नतप्राणाः निशीथाद्वोन्नतासवः ।
> एतद्बाणोनिता त्रिज्या चरज्याढ्या नता यदि ॥
> उन्नताश्चेच्चरज्योना गोले याम्ये विपर्ययात् ।
> द्युज्या लम्बकघातघ्ना त्रिज्याप्ता च पुनर्हृता ॥
> कोटया दृक्क्षेपजीवाया लब्धचापं रवौ क्षिपेत् ।
> तल्लग्नं प्राक्कपाले स्यात् निशि चेत् तद्विवर्जितम् ॥

> The *nata* (hour angle) or *unnata* may be obtained depending upon whether the computation is done at midday or midnight. If *nata* is obtained, then the *bāṇa* (versed sine, *utkramajyā*) of it is to be subtracted from *trijyā* , and the result is added to the *carajyā*. If *unnata*, (is obtained) then the

> result deficient from *carajyā* is found. [This is the procedure for the northern hemisphere]. For the southern hemisphere the operation is reversed.

Here it is essentially stated that

$$R\sin(\lambda_l - \lambda_s) = \frac{R(\cos\phi\cos\delta[\cos H + \sin\Delta\alpha])}{\cos ZV} = \frac{R(\cos\phi\cos\delta\cos H + \sin\phi\sin\delta)}{\cos ZV},$$

where λ_l and λ_s are the *lagna* and the Sun's longitude respectively. From the above equation, taking the inverse sine we get $\lambda_l - \lambda_s$. To this, if we add the longitude of the Sun we get the desired longitude that is *lagna* (λ_l). The rationale behind the above equation can be easily understood with the help of Fig.6.

Applying the cosine formula to the spherical triangle PZS we have,

$$\cos ZS = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H.$$

Similarly from the triangle ZVS we get,

$$\cos VS = \frac{\cos ZS}{\cos ZV}.$$

Now, $VS + SL = VS + \lambda_l - \lambda_s = 90$. Therefore, $\cos VS = \sin(\lambda_l - \lambda_s)$, which results in the above expression for $\sin(\lambda_l - \lambda_s)$.

5 Instantaneous velocity of the true planet

Both in the modern as well as ancient Indian astronomy, the true position of the planet[7] is obtained from the mean position by applying a correction to it. The correction term is known as *mandaphala* , in Indian astronomy, while it is known as 'equation of centre' in modern astronomy.

In Fig.7, A represents the direction of the *mandocca* (apside) and its longitude $\Gamma\hat{O}A = \varpi$. P_0 is the mean planet whose longitude called

[7]Though the treatment is general, in this section planet refers to either the Sun or the Moon.

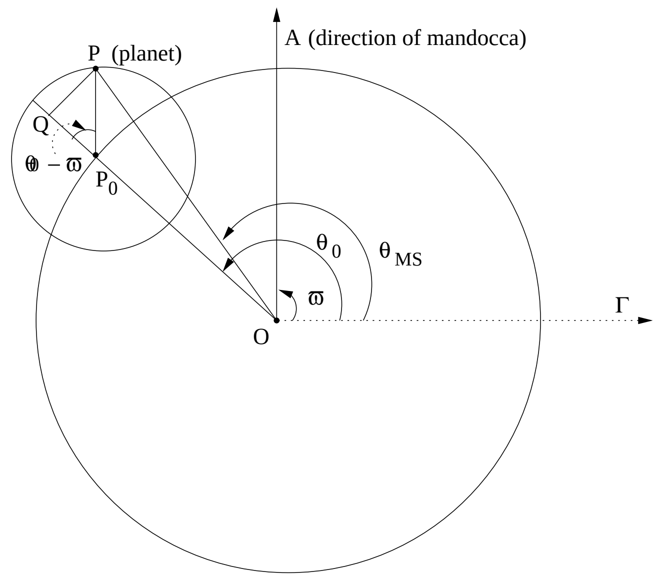

Figure 7: The *manda-samskāra* or the equation of centre correction.

madhyamagraha (mean longitude) is given by $\theta_0 = \Gamma\hat{O}P_0$. The circle of radius r' with P_0 as centre, is the epicycle on which the *mandasphuṭa* (true planet) is located. By construction, PP_0 is parallel to OA. The longitude of the *mandasphuṭa* is given by $\theta_{MS} = \Gamma\hat{O}P$. It can be easily seen that

$$\sin(\theta_0 - \theta_{MS}) = \frac{PQ}{OP} = \frac{r'\sin(\theta_0 - \varpi)}{K},$$

where

$$K = [(R + r'\cos(\theta_0 - \varpi))^2 + r'^2\sin^2(\theta_0 - \varpi)]^{\frac{1}{2}},$$

is the *mandakarṇa*. As per the formulation in *Tantrasaṅgraha* , r' also varies such that

$$\frac{r'}{K} = \frac{r}{R},$$

is a constant. Here r is a given parameter. Then,

$$\sin(\theta_{MS} - \theta_0) = -\frac{r}{R}\sin(\theta_0 - \varpi). \tag{20}$$

The difference between the true (*mandasphuṭa*) and the mean position, is called *mandaphala*, whereas the difference between the *mandocca* and mean position is called *mandakendra*. Denoting them by $\Delta\theta$ and M respectively, the *mandaphala* is given by

$$\Delta\theta = -\sin^{-1}\left(\frac{r}{R}\sin M\right). \tag{21}$$

Thus the *mandasphuṭa* (true longitude) of the planet is given by

$$\theta_{MS} = \theta_0 + \Delta\theta. \tag{22}$$

It may be noted that the calculation of true position of the planet involves sine inverse function (Eq.(22)). Hence, if one needs to find the instantaneous velocity of the planet called *tātkālikagati*, one would have to find the time derivative of this function. It is indeed remarkable that an exact formula for the derivative of sine inverse function is given in *Tantrasaṅgraha* as follows [18]:

चन्द्रबाहुफलवर्गशोधितत्रिज्यकाकृतिपदेन संहरेत् ।
तत्र कोटिफललिप्तिकाहतां केन्द्रभुक्तिरिह यच्च लभ्यते ॥ ५३ ॥
तद्विशोध्य मृगादिके गतेः क्षिप्यतामिह तु कर्कटादिके ।
तद्भवेत्स्फुटतरा गतिर्विधोः अस्य तत्समयजा रवेरपि ॥ ५४ ॥

> Let the product of the *koṭiphala* (in minutes)$[r\cos M]$ and the daily motion of the *manda-kendra* ($\frac{dM}{dt}$) be divided by the square root of the square of the *doḥphala* subtracted from the square of *trijyā* ($\sqrt{R^2 - r^2\sin M}$). The result thus obtained has to be subtracted form the daily motion of the Moon if the *manda-kendra* lies within six signs beginning from *mṛga* and added if it lies within six signs beginning from *karkaṭaka*. The result gives more accurate value of the Moon's angular velocity. In fact, the procedure for finding the instantaneous velocity of the Sun is same as this.

If M be the *manda-kendra* , then the content of the above verse can be expressed in mathematical form as

$$\frac{d}{dt}(\sin^{-1}(\frac{r}{R}\sin M)) = \frac{r\cos M\ \dfrac{dM}{dt}}{\sqrt{R^2 - r^2\sin^2 M}}. \tag{23}$$

This verse appears in the context of finding the true rate of motion of the Moon (instantaneous velocity) from its average rate of motion (mean velocity). The term *gati* refers to the rate of change of the longitude of the planet.

Recalling the expression for the true longitude of the planet, Eq.(22), the rate of change of it is

$$\frac{d}{dt}\theta_{MS} = \frac{d}{dt}\theta_0 - \frac{d}{dt}\Delta\theta. \tag{24}$$

Here, the first term in the RHS represents the mean velocity of the planet and the second term the change in the *mandaphala* given by Eq.(23). In the following section dealing with the computation of the duration of an eclipse, the daily motion of the Moon denoted by d_m is nothing but the derivative of *mandasphuṭa* given by Eq.(23),(24).

6 Algorithm for finding eclipse duration

In Indian astronomy, the total duration of an eclipse is found by determining the first and the second half durations separately. More often than not, there will be significant difference between the two durations. For obtaining accurate values, they are calculated using an iterative procedure, called *aviśeṣakarma*, as will be explained in this section. The sum of the two durations gives the total duration of the eclipse. For instance, if T_1 and T_2 be the first and second half durations of the eclipse, then the total duration of the eclipse is given by

$$T = T_1 + T_2,$$

where both T_1 and T_2 are found iteratively.

6.1 Half duration of the eclipse

The time interval between the Moon entering the umbral portion of the shadow and the instant of opposition is the first half duration, T_1, and that between the instant of opposition and the exit of Moon out of the umbral region is the second half duration T_2. Naively, one may think that these two durations would be equal. However, this is not true because of the continuous change in the velocities of both the Sun and the Moon.

In Fig.6(a), $AX = \beta$ and $OX = S$ represent the latitude of the Moon and the sum of the semi-diameters of the shadow and the Moon respectively. If $d_m - d_s$ refers to the difference in the instantaneous daily motion of the Sun and the Moon, then the first half duration (T_1) is given by

$$T_1 = \frac{OA}{Diff.\ in\ daily\ motion} = \frac{\sqrt{OX^2 - AX^2}}{d_m - d_s} = \frac{\sqrt{S^2 - \beta^2}}{d_m - d_s} \qquad (25)$$

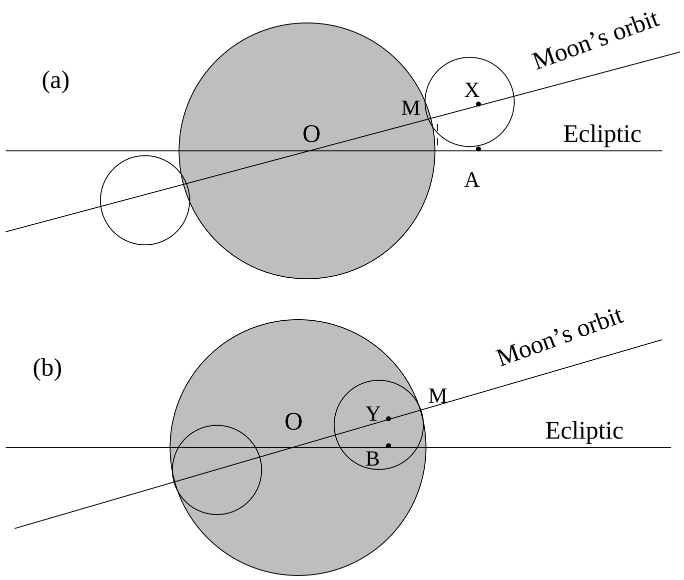

Figure 8: (a) The Earth's shadow and the Moon just before the beginning of the eclipse and just after the release. (b) The Earth's shadow and the Moon just after the beginning of total eclipse and just before its release.

6.2 The need for iteration

In Eq.(25), β is the Moon's latitude at the beginning of the eclipse. The denominator represents the difference in the daily motion of the Sun and the Moon. If λ_m and λ_s are the longitudes of the Sun and the Moon, this difference in their daily motion called *gatyantara* or *bhuktyantara* is given by

$$gatyantara = D(t) = \frac{d}{dt}(\lambda_m - \lambda s).$$

Initially, β and $D(t)$ are calculated at the instant of opposition. If Moon's latitude and the rate of motion of the Sun and the Moon were to be constant, then Eq.(25) would at once give the correct half duration of the eclipse and there would be no need for an iterative procedure. However, they are continuously varying quantities. Hence, determining T_1, using the latitude and *gatyantara* computed at the instant of opposition is only approximate and to get more accurate values *aviśeṣakarma*, a special kind of iterative procedure, is prescribed.

6.3 Concept of *Aviśeṣakarma*

Aviśeṣakarma refers to an iterative process that is to be carried out whenever there is an interdependency (*anyonyāśraya*) between the quantity to be calculated and the parameters which are involved in the calculation. For instance, in eclipse calculation, without knowing the latitude at the end of the eclipse, we will not be able to give the exact half duration, and without knowing the half-duration it is not possible to find the exact latitude of the Moon. To get over this tricky situation, *aviśeṣakarma* is recommended. The need for it is succinctly explained Śaṅkara Vāriyar in his *Yuktidīpikā* as follows [19]:

> विक्षेपकोटिवृत्तं तत् नानारूपं प्रतिक्षणम् ।
> क्षेपो भिन्नो भवेत् स्पर्शमध्यमोक्षेषु शीतगोः ॥
> स्थित्यर्धं द्वितयं भिन्नं क्षेपे भिन्ने भवेत् ततः ।
> ततोऽविशेषविधिना स्थियर्धद्वयमानयेत् ॥

> The declination circle (of the Moon) keeps changing every second (continuously). [Hence], the latitudinal deflection of the Moon is different at the beginning, middle and the end of the eclipse.
>
> Thus the two half durations (first and the second half) differ because of the variation in the latitudinal deflection. There-

> fore, the half-duration may be determined through the *aviśeṣa* process.

The word *viśeṣa* means 'distinction'; therefore *aviśeṣa* is "without distinction". Though the meanings of the words *viśeṣa* and *aviśeṣa* are opposed to each other, the latter should not be taken to refer to *tulya* or 'completely identical'. In the context of mathematical calculations, it only means – *without distinction to a desired degree of accuracy.* In other words, in *aviśeṣakarma*, the iterative process needs to be carried out only up to a point wherein the two successive values of the results are 'without distinction' for a desired degree of accuracy. Once this accuracy is reached the process should be terminated.

6.4 The iterative process for half duration

Now, we illustrate the *aviśeṣa* process by considering the example of finding the half duration of an eclipse. Let t_m be the time of opposition or *madhyakāla* of a lunar eclipse, and Δt_0 be the zeroth order approximation of the half duration of eclipse determined with the parameter values obtained at t_m. That is,

$$\Delta t_0 = \frac{\sqrt{S^2(t_m) - \beta^2(t_m)}}{D(t_m)}$$

To get the first approximation, the sum of the semi diameters S, the latitude of the Moon β and the difference in daily motion D are then determined at $t_m - \Delta t_0$. With them the first approximation to the half duration is obtained. It is given by

$$\Delta t_1 = \frac{\sqrt{S^2(t_m - \Delta t_0) - \beta^2(t_m - \Delta t_0)}}{D(t_m - \Delta t_0)}$$

To get the second approximation, S, β and D are determined at $t_m - \Delta t_1$. With these values the second approximation to the half duration is

$$\Delta t_2 = \frac{\sqrt{S^2(t_m - \Delta t_1) - \beta^2(t_m - \Delta t_1)}}{D(t_m - \Delta t_1)}$$

Similarly, determining S, β and D at $t_m - \Delta t_2$, the third approximation is found.

$$\Delta t_3 = \frac{\sqrt{S^2(t_m - \Delta t_2) - \beta^2(t_m - \Delta t_2)}}{D(t_m - \Delta t_2)}$$

This process will be continued until,

$$\Delta t_n - \Delta t_{n-1} < \epsilon, \tag{26}$$

where ϵ is the desired degree of accuracy. At this stage, since Δt has converged to the desired accuracy, the iteration is terminated. In terms of the notation used earlier, the first half duration $T_1 = \Delta t_n$. Therefore, the instant of the commencement of the eclipse, known as *sparśakāla*, is given by

$$t_b = t_m - \Delta t_n. \tag{27}$$

A similar procedure is to be adopted for the determination of *mokṣakāla* (t_e), with the only difference that, instead of subtracting, the half duration $\Delta t'$ from t_m, we need to add to it. If $\Delta t'_i$ be the second half duration of the eclipse obtained after the i^{th} iteration, then it is given by

$$\Delta t'_i = \frac{\sqrt{S^2(t_m + \Delta t_{i-1}) - \beta^2(t_m + \Delta t_{i-1})}}{D(t_m + \Delta t_{i-1})}$$

As in the case of *sparśa*, here again the process of iteration has to be continued till $\Delta t'_i$ converges. That is,

$$\Delta t'_r - \Delta t'_{r-1} < \epsilon. \tag{28}$$

At this stage, the second half duration of the eclipse and the *mokṣakāla* are given by

$$\begin{aligned} T_2 &= \Delta t'_r \\ t_e &= t_m + \Delta t'_r \end{aligned} \tag{29}$$

Now, the total duration of the eclipse is $t_e - t_b = T_1 + T_2$.

7 Concluding Remarks

Many of the algorithms presented in the paper, barring some refinements, can be found even in the celebrated text *Āryabhaṭīya* , composed by Āryabhaṭa as early as 499 AD. The Kerala school of astronomy and mathematics which is well known for its pioneering work in mathematical analysis and many innovations in the Indian astronomical tradition has tried to perfect these algorithms, the culmination of which can be seen in the works of Nīlakaṇṭha Somayājī. Particularly, the exact algorithm for the computation of *lagna* and the formula for the instantaneous velocity of the planet presented by Nīlakaṇṭha , are indicative of how in the Indian astronomical tradition there has been a continuous endeavour to improvise and achieve better and better accuracy in all computations. As regards the *aviśeṣakarma*, it will be interesting to study the convergence properties of this iterative process as employed in different contexts in Indian Astronomy.

Acknowledgement

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2. C.B.Boyer, *A History of Mathematics*, John Wiley and Sons, 1989, p 256.

3. See for instance, *500 years of Tantrasaṅgraha : A Landmark in the History of Astronomy*, Ed. by M.S.Sriram, K.Ramasubramanian and M.D.Srinivas, Indian Institute of Advanced Study, Shimla, 2002.

4. (i) The verses attributed to Mādhava (c.14th century) by Śaṅkara Vāriyar in his commentary to *Tantrasaṅgraha* (Chap 2, verses 437, 438) beginning with विद्वान् तुन्नबल ``` and स्तेनः स्त्रीपिशुनः ```, for

obtaining the sine and cosine values for ANY DESIRED ANGLE, yield results correct up to 7 decimal places. This is indeed a remarkable result which may be considered far ahead of his times. (ii) For more details and mathematical exposition of the above the reader may refer to the article by M.S. Sriram published in the present volume.

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9. See for instance, *Siddhantaśiromaṇi* , Chap 3, verses 12-14.

10. For a detailed exposition on this, see K.Ramasubramanian and M.S.Sriram, *Corrections to the terrestrial latitude in Tantrasaṅgraha* , Indian Journal of History of Science, **38.2**, 2003, p 129-144.

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13. *ibid.*, verses 95-100.

14. *ibid.*, verses 100-101.

15. *ibid.*, verse 102.

16. *ibid.*, verses 104-105.

17. *ibid.*, verses 107-109.

18. *ibid.*, Chapter 2, verses 53-54.

19. *Tantrasaṅgraha* of Nīlakaṇṭha with *Yuktidīpikā* , commentary in the form of verses by Śaṅkara Vāriyar, Ed. by K.V.Sarma, VVBIS, Punjab University, 1977, Chap 4, verses 79 and 87, p 261-62.

Proofs in Indian Mathematics

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Abstract

Contrary to the widespread belief that Indian mathematicians did not present any proofs for their results, it is indeed the case that there is a large body of source-works in the form of commentaries which present detailed demonstrations (referred to as *upapatti*-s or *yukti*-s) for the various results enunciated in the major texts of Indian Mathematics and Astronomy. Amongst the published works, the earliest exposition of *upapatti*-s are to be found in the commentaries of Govindasvāmin (c.800) and Caturveda Pṛthūdakasvāmin (c.860). Then we find very detailed exposition of *upapatti*-s in the works of Bhāskarācārya II (c.1150). In the medieval period we have the commentaries of Śaṅkara Vāriyar (c.1535), Gaṇeśa Daivajña (c.1545), Kṛṣṇa Daivajña (c.1600) and the famous Malayalam work *Yuktibhāṣā* of Jyeṣṭhadeva (c.1530), which present detailed *upapatti*-s. By presenting a few selected examples of *upapatti*-s, we shall highlight the logical rigour which is characteristic of all the work in Indian Mathematics. We also discuss how the notion of *upapatti* is perhaps best understood in the larger epistemological perspective provided by *Nyāyaśāstra*, the Indian School of Logic. This could be of help in explicating some of the important differences between the notion of *upapatti* and the notion of "proof" developed in the Greco-European tradition of Mathematics.

1 Alleged Absence of Proofs in Indian Mathematics

Several books have been written on the history of Indian tradition in mathematics.[1] In addition, many books on history of mathematics devote a section, sometimes even a chapter, to the discussion of Indian mathematics. Many of the results and algorithms discovered by the Indian mathematicians have been studied in some detail. But, little attention has been paid to the methodology and foundations of Indian mathematics. There is hardly any discussion of the processes by which Indian mathematicians arrive at and justify their results and procedures. And, almost no attention is paid to the philosophical foundations of Indian mathematics, and the Indian understanding of the nature of mathematical objects, and validation of mathematical results and procedures.

Many of the scholarly works on history of mathematics assert that Indian Mathematics, whatever its achievements, does not have any sense of logical rigour. Indeed, a major historian of mathematics presented the following assessment of Indian mathematics over fifty years ago:

> The Hindus apparently were attracted by the arithmetical and computational aspects of mathematics rather than by the geometrical and rational features of the subject which had appealed so strongly to the Hellenistic mind. Their name for mathematics, *gaṇita*, meaning literally the 'science of calculation' well characterizes this preference. They delighted more in the tricks that could be played with numbers than in the thoughts the mind could produce, so that neither Euclidean geometry nor Aristotelian logic made a strong impression upon them. The Pythagorean problem of the incommensurables, which was of intense interest to Greek geometers, was of little import to Hindu mathematicians, who treated rational and irrational quantities, curvilinear

[1] We may cite the following standard works: B.B.Datta and A.N.Singh, *History of Hindu Mathematics,* 2 parts, Lahore 1935, 1938, Reprint, Delhi 1962; C.N.Srinivasa Iyengar, *History of Indian Mathematics,* Calcutta 1967; A.K.Bag, *Mathematics in Ancient and Medieval India,* Varanasi 1979; T.A.Saraswati Amma, *Geometry in Ancient and Medieval India*, Varanasi 1979; G.C.Joseph, *The Crest of the Peacock: The Non-European Roots of Mathematics,* 2^{nd} Ed., Princeton 2000.

> and rectilinear magnitudes indiscriminately. With respect to the development of algebra, this attitude occasioned perhaps an incremental advance, since by the Hindus the irrational roots of the quadratics were no longer disregarded as they had been by the Greeks, and since to the Hindus we owe also the immensely convenient concept of the absolute negative. These generalizations of the number system and the consequent freedom of arithmetic from geometrical representation were to be essential in the development of the concepts of calculus, but the Hindus could hardly have appreciated the theoretical significance of the change...
>
> The strong Greek distinction between the discreteness of number and the continuity of geometrical magnitude was not recognized, for it was superfluous to men who were not bothered by the paradoxes of Zeno or his dialectic. Questions concerning incommensurability, the infinitesimal, infinity, the process of exhaustion, and the other inquiries leading toward the conceptions and methods of calculus were neglected.[2]

Such views have found their way generally into more popular works on history of mathematics. For instance, we may cite the following as being typical of the kind of opinions commonly expressed about Indian mathematics:

> As our survey indicates, the Hindus were interested in and contributed to the arithmetical and computational activities of mathematics rather than to the deductive patterns. Their

[2] C.B.Boyer, *The History of Calculus and its Conceptual development,* New York 1949, p.61-62. As we shall see in the course of this article, Boyer's assessment – that the Indian mathematicians did not reach anywhere near the development of calculus or mathematical analysis, because they lacked the sophisticated methodology developed by the Greeks – seems to be thoroughly misconceived. In fact, in marked contrast to the development of mathematics in the Greco-European tradition, the methodology of Indian mathematical tradition seems to have ensured continued and significant progress in all branches of mathematics till barely two hundred year ago; it also lead to major discoveries in calculus or mathematical analysis, without in anyway abandoning or even diluting its standards of logical rigour, so that these results, and the methods by which they were obtained, seem as much valid today as at the time of their discovery.

> name for mathematics was *gaṇita*, which means "the science of calculation". There is much good procedure and technical facility, but no evidence that they considered proof at all. They had rules, but apparently no logical scruples. Moreover, no general methods or new viewpoints were arrived at in any area of mathematics.
>
> It is fairly certain that the Hindus did not appreciate the significance of their own contributions. The few good ideas they had, such as separate symbols for the numbers from 1 to 9, the conversion to base 10, and negative numbers, were introduced casually with no realization that they were valuable innovations. They were not sensitive to mathematical values. Along with the ideas they themselves advanced, they accepted and incorporated the crudest ideas of the Egyptians and Babylonians.[3]

The burden of scholarly opinion is such that even eminent mathematicians, many of whom have had fairly close interaction with contemporary Indian mathematics, have ended up subscribing to similar views, as may be seen from the following remarks of one of the towering figures of twentieth century mathematics:

> For the Indians, of course, the effectiveness of the *cakravāla* could be no more than an experimental fact, based on their treatment of great many specific cases, some of them of considerable complexity and involving (to their delight, no doubt) quite large numbers. As we shall see, Fermat was the first one to perceive the need for a general proof, and Lagrange was the first to publish one. Nevertheless, to have developed the *cakravāla* and to have applied it successfully to such difficult numerical cases as $N = 61$, or $N = 67$ had been no mean achievements.[4]

[3]Morris Kline, *Mathematical Thought from Ancient to Modern Times,* Oxford 1972, p.190.

[4]Andre Weil, *Number Theory: An Approach through History from Hammurapi to Legendre,* Boston 1984, p.24. It is indeed ironical that Prof. Weil has credited Fermat, who is notorious for not presenting proofs for most of the claims he made, with the realization that mathematical results need to be justified by proofs. While the rest of this article is purported to show that the Indian mathematicians presented logically

Modern scholarship seems to be unanimous in holding the view that Indian mathematics is bereft of any notion of proof. But even a cursory study of the source-works that are available in print would reveal that Indian mathematicians place much emphasis on providing what they refer to as *upapatti* (proof, demonstration) for every one of their results and procedures. Some of these *upapatti*-s were noted in the early European studies on Indian mathematics in the first half of the nineteenth century. For instance, in 1817, H.T. Colebrooke notes the following in the preface to his widely circulated translation of portions of *Brāhmasphuṭasiddhānta* of Brahmagupta and *Līlāvatī* and *Bījagaṇita* of Bhāskarācārya:

> On the subject of demonstrations, it is to be remarked that the Hindu mathematicians proved propositions both algebraically and geometrically: as is particularly noticed by Bhāskara himself, towards the close of his algebra, where he gives both modes of proof of a remarkable method for the solution of indeterminate problems, which involve a factum of two unknown quantities.[5]

Another notice of the fact that detailed proofs are provided in the Indian texts on mathematics is due to C.M.Whish who, in an article published in 1835, pointed out that infinite series for π and for trigonometric functions were derived in texts of Indian mathematics much before their 'discovery' in Europe. Whish concluded his paper with a sample proof

rigorous proofs for most of the results and processes that they discovered, it must be admitted that the particular example that Prof. Weil is referring to, the effectiveness of the *cakravāla* algorithm (known to the Indian mathematicians at least from the time of Jayadeva, prior to the eleventh century) for solving quadratic indeterminate equations of the form $x^2 - Ny^2 = 1$, does not seem to have been demonstrated in the available source-works. In fact, the first proof of this result was given by Krishnaswamy Ayyangar barely seventy-five years ago (A.A.Krishnaswamy Ayyangar, "New Light on Bhāskara's *Cakravāla* or Cyclic Method of solving Indeterminate Equations of the Second Degree in Two Variables', Jour. Ind. Math. Soc. **18**, 228-248, 1929-30). Krishnaswamy Ayyangar also showed that the *cakravāla* algorithm is different and more optimal than the Brouncker-Wallis-Euler-Lagrange algorithm for solving this so-called "Pell's Equation."

[5]H.T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhāskara,* London 1817, p.xvii. Colebrooke also presents some of the *upapatti*-s given by the commentators Gaṇeśa Daivajña and Kṛṣṇa Daivajña, as footnotes in his work.

from the Malayalam text *Yuktibhāṣā* of the theorem on the square of the hypotenuse of a right angled triangle and also promised that:

> A further account of the *Yuktibhāṣā*, the demonstrations of the rules for the quadrature of the circle of infinite series, with the series for the sines, cosines, and their demonstrations, will be given in a separate paper: I shall therefore conclude this, by submitting a simple and curious proof of the 47^{th} proposition of Euclid [the so called Pythagoras theorem], extracted from the *Yuktibhāṣā*.[6]

It would indeed be interesting to find out how the currently prevalent view, that Indian mathematics lacks the notion of proof, obtained currency in the last 100-150 years.

2 *Upapatti*-s in Indian Mathematics

2.1 The tradition of *Upapatti*-s in Mathematics and Astronomy

A major reason for our lack of comprehension, not merely of the Indian notion of proof, but also of the entire methodology of Indian mathematics, is the scant attention paid to the source-works so far. It is said that there are over one hundred thousand manuscripts on *Jyotiḥśāstra*, which includes, apart from works on *gaṇita* (mathematics and mathematical astronomy), also those on *saṃhitā* (omens) and *hora* (astrology).[7] Only a small fraction of these texts have been published. A well known source book, lists about 285 published works in mathematics and mathematical astronomy. Of these, about 50 are from the period before 12^{th} century AD, about 75 from $12^{th} - 15^{th}$ centuries, and about 165 from $16^{th} - 19^{th}$

[6] C.M. Whish, 'On the Hindu Quadrature of the Circle, and the Infinite Series of the Proportion of the Circumference to the Diameter Exhibited in the Four Shastras, the Tantrasangraham, Yucti Bhasa, Carana Paddhati and Sadratnamala', Trans.Roy.As.Soc.(G.B.) **3**, 509-523, 1835. However, Whish does not seem to have published any further paper on this subject.

[7] D. Pingree, *Jyotiḥśāstra*: *Astral and Mathematical Literature*, Wiesbaden 1981, p.118.

centuries.[8]

Much of the methodological discussion is usually contained in the detailed commentaries; the original works rarely touch upon such issues. Modern scholarship has concentrated on translating and analysing the original works alone, without paying much heed to the commentaries. Traditionally, the commentaries have played at least as great a role in the exposition of the subject as the original texts. Great mathematicians and astronomers, of the stature of Bhāskarācārya I, Bhāskarācārya II, Parameśvara, Nīlakaṇṭha Somasutvan, Gaṇeśa Daivajña, Munīśvara and Kamalākara, who wrote major original treatises of their own, also took great pains to write erudite commentaries on their own works and on the works of earlier scholars. It is in these commentaries that one finds detailed *upapatti*-s of the results and procedures discussed in the original texts, as also a discussion of the various methodological and philosophical issues. For instance, at the beginning of his commentary *Buddhivilāsinī*, Gaṇeśa Daivajña states:

> श्रीभास्करोक्तवचसामपि संस्फुटानां
> व्याख्याविशेषकथनेन न चास्ति चित्रम् ।
> अत्रोपपत्तिकथनेऽखिलसारभूते
> पश्यन्तु सुज्ञगणका मम बुद्धिचित्रम् ॥

> There is no purpose served in providing further explanations for the already lucid statements of Śrī Bhāskara. The knowledgeable mathematicians may therefore note the specialty of my intellect in the statement of *upapatti*-s, which are after all the essence of the whole thing.[9]

Amongst the published works on Indian mathematics and astronomy, the earliest exposition of *upapatti*-s are to be found in the *bhāṣya* of Govindasvāmin (c 800) on *Mahābhāskarīya* of Bhāskarācārya I, and the *Vāsanābhāṣya* of Caturveda Pṛthūdakasvāmin (c 860) on *Brāhma-*

[8] K.V. Sarma and B.V. Subbarayappa, *Indian Astronomy: A Source Book,* Bombay 1985.

[9] *Buddhivilāsinī* of Gaṇeśa Daivajña, V.G. Apte (ed.), Vol I, Pune 1937, p.3.

sphuṭasiddhānta of Brahmagupta.[10] Then we find very detailed exposition of *upapatti*-s in the works of Bhāskarācārya II (c.1150): his *Vivaraṇa* on *Śiṣyadhīvṛddhidatantra* of Lalla and *Vāsanābhāṣya* on his own *Siddhāntaśiromaṇi*.[11] Apart from these, Bhāskarācārya provides an idea of what is an *upapatti* in his *Bījavāsanā* on his own *Bījagaṇita* in two places. In the chapter on *madhyamāharaṇa* (quadratic equations) he poses the following problem:

> Find the hypotenuse of a plane figure, in which the side and upright are equal to fifteen and twenty. And show the *upapatti* (demonstration) of the standard procedure of computation.[12]

Bhāskarācārya provides two *upapatti*-s for the solution of this problem, the so-called Pythagoras theorem; and we shall consider them later. Again, towards the end of the *Bījagaṇita* in the chapter on *bhāvita* (equations involving products), while considering integral solutions of equations of the form $ax + by = cxy$, Bhāskarācārya explains the nature of *upapatti* with the help of an example:

> The *upapatti* (demonstration) follows. It is twofold in each case: One geometric and the other algebraic. The geometric demonstration is here presented... The algebraic demonstration is next set forth... This procedure [of demonstration] has been earlier presented in a concise instructional form [*saṃkṣiptapāṭha*] by ancient teachers. The algebraic demonstrations are for those who do not comprehend the geometric one. Mathematicians have said that algebra is computation joined with demonstration; otherwise there would be no difference between arithmetic and algebra. Therefore this demonstration of *bhāvita* has been shown in two ways.[13]

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[10] The *Āryabhaṭīya-bhāṣya* of Bhāskara I (c.629) does occasionally indicate the derivation of some of the mathematical procedures, though his commentary does not purport to present *upapatti*-s for the rules and procedures given in *Āryabhaṭīya*.

[11] This latter commentary of Bhāskara II is a classic source of *upapatti*-s and needs to be studied in depth.

[12] *Bījagaṇita* of Bhāskarācārya, Muralidhara Jha (ed.), Varanasi 1927, p.69.

[13] *Bījagaṇita*, cited above, p.125-127.

Clearly the tradition of exposition of *upapatti*-s is much older and Bhāskarācārya, and the later mathematicians and astronomers are merely following the traditional practice of providing detailed *upapatti*-s in their commentaries to earlier, or their own, works.[14]

In Appendix A we give a list of important commentaries, available in print, which present detailed *upapatti*-s. It is unfortunate that none of the published source-works that we have mentioned above has so far been translated into any of the Indian languages, or into English; nor have they been studied in depth with a view to analyze the nature of mathematical arguments employed in the *upapatti*-s or to comprehend the methodological and philosophical foundations of Indian mathematics and astronomy.[15]

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[14]Ignoring all these classical works on *upapatti*-s, one scholar has recently claimed that the tradition of *upapatti* in India "dates from the 16^{th} and 17^{th} centuries" (J.Bronkhorst, '*Pāṇini* and Euclid', Jour. Ind. Phil. **29**, 43-80, 2001).

[15]We may, however, mention the following works of C.T.Rajagopal and his collaborators which discuss some of the *upapatti*-s presented in the Malayalam work *Yuktibhāṣā* of Jyeṣṭhadeva (c.1530) for various results in geometry, trigonometry and those concerning infinite series for π and the trigonometric functions: K. Mukunda Marar, 'Proof of Gregory's Series', Teacher's Magazine **15**, 28-34, 1940; K. Mukunda Marar and C.T.Rajagopal, 'On the Hindu Quadrature of the Circle', J.B.B.R.A.S. **20**, 65-82, 1944; K. Mukunda Marar and C.T.Rajagopal, 'Gregory's Series in the Mathematical Literature of Kerala', Math Student **13**, 92-98, 1945; A. Venkataraman, 'Some Interesting Proofs from *Yuktibhāṣā* ', Math Student **16**, 1-7, 1948; C.T.Rajagopal 'A Neglected Chapter of Hindu Mathematics', Scr. Math. **15**, 201-209, 1949; C.T.Rajgopal and A. Venkataraman, 'The Sine and Cosine Power Series in Hindu Mathematics', J.R.A.S.B. **15**, 1-13, 1949; C.T. Rajagopal and T.V.V.Aiyar, 'On the Hindu Proof of Gregory's Series', Scr. Math. **17**, 65-74, 1951; C.T.Rajagopal and T.V.V.Aiyar, 'A Hindu Approximation to Pi', Scr.Math. **18**, 25-30, 1952. C.T.Rajagopal and M.S.Rangachari, 'On an Untapped Source of Medieval Keralese Mathematics', Arch. for Hist. of Ex. Sc. **18**, 89-101, 1978; C.T.Rajagopal and M.S.Rangachari, 'On Medieval Kerala Mathematics', Arch. for Hist. of Ex. Sc. **35**(2), 91-99, 1986.

Following the work of Rajagopal and his collaborators, there are some recent studies which discuss some of the proofs in *Yuktibhāṣā* . We may here cite the following: T.Hayashi, T.Kusuba and M.Yano, 'The Correction of the Mādhava Series for the Circumference of a Circle', Centauras, **33**, 149-174, 1990; Ranjan Roy, 'The Discovery of the Series formula for π by Leibniz, Gregory and Nīlakaṇṭha', Math. Mag. **63**, 291-306, 1990; V.J.Katz, 'Ideas of Calculus in Islam and India', Math. Mag. **68**, 163-174, 1995; C.K.Raju, 'Computers, Mathematics Education, and the Alternative Epistemology of the Calculus in the *Yuktibhāṣā* ', Phil. East and West **51**, 325-362, 2001; D.F.Almeida, J.K.John and A.Zadorozhnyy, 'Keralese Mathematics: Its Possible Transmission to Europe and the Consequential Educational Implications', In this article, we shall present some examples of the kinds of *upapatti*-s provided in Indian mathematics, from the commentaries of Gaṇeśa Daivajña (c.1545) and Kṛṣṇa Daivajña (c.1600) on the texts *Līlāvatī* and *Bījagaṇita* respectively, of Bhāskarācārya II (c.1150), and from the celebrated Malayalam work *Yuktibhāṣā* of Jyeṣṭhadeva (c.1530). We shall also discuss how the notion of *upapatti* is perhaps best understood in the larger epistemological perspective provided by *Nyāya-śāstra* the Indian School of Logic. This enables us to explicate some of the important differences between the notion of *upapatti* and the notion of "proof" developed in the Greco-European tradition of Mathematics.

2.2 Mathematical results should be supported by *Upapatti*-s

Before discussing some of the *upapatti*-s presented in Indian mathematical tradition, it is perhaps necessary to put to rest the widely prevalent myth that the Indian mathematicians did not pay any attention to, and perhaps did not even recognize the need for justifying the mathematical results and procedures that they employed. The large corpus of *upapatti*-s, even amongst the small sample of source-works published so far, should convince anyone that there is no substance to this myth. Still, we may cite the following passage from Kṛṣṇa Daivajña's commentary *Bījapallava* on *Bījagaṇita* of Bhāskarācārya, which clearly brings out the basic understanding of Indian mathematical tradition that citing any number of instances (even an infinite number of them) where a particular result seems to hold, does not amount to establishing that as a valid result in mathematics; only when the result is supported by a *upapatti* or a demonstration, can the result be accepted as valid:

> How can we state without proof (*upapatti*) that twice the product of two quantities when added or subtracted from

J. Nat. Geo. **20**, 77-104, 2001; D.Bressoud, 'Was Calculus Invented in India?', College Math. J. **33**, 2-13, 2002; J.K.John, 'Derivation of the *Saṃskāras* applied to the Mādhava Series in *Yuktibhāṣā* ', in M.S.Sriram, K.Ramasubramanian and M.D.Srinivas (eds.), *500 Years of Tantrasaṅgraha : A Landmark in the History of Astronomy*, Shimla 2002, p 169-182. An outline of the proofs given in *Yuktibhāṣā* can also be found in T.A. Saraswati Amma, 1979, cited earlier, and in S.Parameswaran, *The Golden Age of Indian Mathematics,* Kochi 1998.

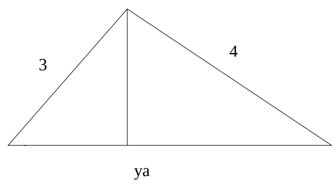
the sum of their squares is equal to the square of the sum or difference of those quantities? That it is seen to be so in a few instances is indeed of no consequence. Otherwise, even the statement that four times the product of two quantities is equal to the square of their sum, would have to be accepted as valid. For, that is also seen to be true in some cases. For instance, take the numbers 2, 2. Their product is 4, four times which will be 16, which is also the square of their sum 4. Or take the numbers 3, 3. Four times their product is 36, which is also the square of their sum 6. Or take the numbers 4, 4. Their product is 16, which when multiplied by four gives 64, which is also the square of their sum 8. Hence, the fact that a result is seen to be true in some cases is of no consequence, as it is possible that one would come across contrary instances (*vyabhicāra*) also. Hence it is necessary that one would have to provide a proof (*yukti*) for the rule that twice the product of two quantities when added or subtracted from the sum of their squares results in the square of the sum or difference of those quantities. We shall provide the proof (*upapatti*) in the end of the section on *ekavarṇa-madhyamāharaṇa*.[16]

2.3 Square of the hypotenuse of a right-angled triangle: the so-called Pythagoras Theorem

Gaṇeśa provides two *upapatti*-s for the rule concerning the square of the hypotenuse (*karṇa*) of a right-angled triangle.[17] These *upapatti*-s are the same as the ones outlined by Bhāskarācārya II in his *Bījavāsanā* on his own *Bījagaṇita*, that we referred to earlier. The first involves the *avyakta* method and proceeds as follows:

[16] *Bījapallava* of Kṛṣṇa Daivajña, T.V. Radhakrishna Sastri (ed.), Tanjore, 1958, p.54.

[17] *Buddhivilāsinī*, cited earlier, p.128-129.



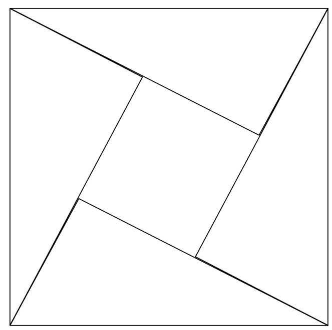
$$y\bar{a} = \left(\frac{9}{y\bar{a}}\right) + \left(\frac{16}{y\bar{a}}\right)$$
$$y\bar{a}^2 = 25$$
$$y\bar{a} = 5$$

Take the hypotenuse (*karṇa*) as the base and assume it to be *yā*. Let the *bhujā* and *koṭi* (the two sides) be 3 and 4 respectively. Take the hypotenuse as the base and draw the perpendicular to the hypotenuse from the opposite vertex as in the figure. [This divides the triangle into two triangles, which are similar to the original] Now by the rule of proportion (*anupāta*), if *yā* is the hypotenuse the *bhujā* is 3, then when this *bhujā* 3 is the hypotenuse, the *bhujā*, which is now the *ābādhā* (segment of the base) on the side of the original *bhujā* will be $(\frac{9}{y\bar{a}})$.

Again if *yā* is the hypotenuse, the *koṭi* is 4, then when this *koṭi* 4 is the hypotenuse, the *koṭi*, which is now the segment of base on the side of the (original) *koṭi* will be $(\frac{16}{y\bar{a}})$. Adding the two segments (*ābādhā*-s) of *yā* the hypotenuse and equating the sum to (the hypotenuse) *yā*, cross-multiplying and taking the square-roots, we get $y\bar{a} = 5$, the square root of the sum of the squares of *bhujā* and *koṭi* .

The other *upapatti* of Gaṇeśa is *kṣetragata* or geometrical, and proceeds as follows:[18]

[18]This method seems to be known to Bhāskarācārya I (c.629 AD) who gives a very similar diagram in his *Āryabhaṭīyabhāṣya*, K.S. Shukla (ed.), Delhi 1976, p.48. The Chinese mathematician Liu Hui (c 3^{rd} century AD) seems to have proposed similar geometrical proofs of this so-called Pythagoras Theorem. See for instance, D.B.Wagner, 'A Proof of the Pythagorean Theorem by Liu Hui', Hist. Math.**12**, 71-3, 1985.



$$c^2 = (a-b)^2 + 4(\tfrac{1}{2}ab) = a^2 + b^2$$

Take four triangles identical to the given and taking the four hypotenuses to be the four sides, form the square as shown. Now, the interior square has for its side the difference of *bhujā* and *koṭi* . The area of each triangle is half the product of *bhujā* and *koṭi* and four times this added to the area of the interior square is the area of the total figure. This is twice the product of *bhujā* and *koṭi* added to the square of their difference. This, by the earlier cited rule, is nothing but the sum of the squares of *bhujā* and *koṭi* . The square root of that is the side of the (big) square, which is nothing but the hypotenuse.

2.4 The rule of signs in Algebra

One of the important aspects of Indian mathematics is that in many *upapatti*-s the nature of the underlying mathematical objects plays an important role. We can for instance, refer to the *upapatti* given by Kṛṣṇa Daivajña for the well-known rule of signs in Algebra. While providing an *upapatti* for the rule, "the number to be subtracted if positive (*dhana*) is made negative (*ṛṇa*) and if negative is made positive", Kṛṣṇa Daivajña states:

> Negativity (*ṛṇatva*) here is of three types – spatial, temporal and that pertaining to objects. In each case, it [negativity] is indeed the *vaiparītya* or the oppositeness. . .For instance, the other direction in a line is called the opposite direction (*viparīta dik*); just as west is the opposite of east. . . Further, between two stations if one way of traversing is considered positive then the other is negative. In the same way, past and future time intervals will be mutually negative

> of each other... Similarly, when one possesses said objects they would be called his *dhana* (wealth). The opposite would be the case when another owns the same objects... Amongst these [different conceptions], we proceed to state the *upapatti* of the above rule, assuming positivity (*dhanatva*) for locations in the eastern direction and negativity (*ṛṇatva*) for locations in the west, as follows... [19]

Kṛṣṇa Daivajña goes on to explain how the distance between a pair of stations can be computed knowing that between each of these stations and some other station on the same line. Using this he demonstrates the above rule that "the number to be subtracted if positive is made negative..."

2.5 The *Kuṭṭaka* process for the solution of linear indeterminate equations

To understand the nature of *upapatti* in Indian mathematics one will have to analyse some of the lengthy demonstrations which are presented for the more complicated results and procedures. One will also have to analyse the sequence in which the results and the demonstrations are arranged to understand the method of exposition and logical sequence of arguments. For instance, we may refer to the demonstration given by Kṛṣṇa Daivajña[20] of the well-known *kuṭṭaka* procedure, which has been employed by Indian mathematicians at least since the time of Āryabhaṭa (c 499 AD), for solving first order indeterminate equations of the form

$$\frac{(ax+c)}{b} = y,$$

where a, b, c are given integers and x, y are to be solved for in integers. Since this *upapatti* is rather lengthy, we merely recount the essential steps here.[21]

[19] *Bījapallava*, cited above, p.13.
[20] *Bījapallava*, cited above, p.85-99.
[21] A translation of the *upapatti* may be found in M.D.Srinivas, 'Methodology of Indian Mathematics and its Contemporary Relevance', PPST Bulletin, **12**, 1-35, 1987.

Kṛṣṇa Daivajña first shows that the solutions for x, y do not vary if we factor all the three numbers $a,\ b,\ c$ by the same common factor. He then shows that if a and b have a common factor, then the above equation will not have a solution unless c is also divisible by the same common factor. Then follows the *upapatti* of the process of finding the greatest common factor of a and b by mutual division, the so-called Euclidean algorithm. He then provides an *upapatti* for the *kuṭṭaka* method of finding the solution which involves carrying out a sequence of transformations on the *vallī* (line or column) of quotients obtained in the above mutual division. This is based on a detailed analysis of the various operations in reverse (*vyasta-vidhi*). The last two elements of the *vallī*, at each stage, are shown to be the solutions of the *kuṭṭaka* problem involving the successive pair of remainders (taken in reverse order from the end) which arise in the mutual division of a and b. Finally, it is shown how the procedure differs depending upon whether there are odd or even number of coefficients generated in the above mutual division.

2.6 Nīlakaṇṭha's proof for the sum of an infinite geometric series

In his *Āryabhaṭīyabhāṣya* while deriving an interesting approximation for the arc of circle in terms of the *jyā* (Rsine) and the *śara* (Rversine), the celebrated Kerala astronomer Nīlakaṇṭha Somasutvan presents a detailed demonstration of how to sum an infinite geometric series. Though it is quite elementary compared to the various other infinite series expansions derived in the works of the Kerala School, we shall present an outline of Nīlakaṇṭha's argument as it clearly shows how the notion of limit was well understood in the Indian mathematical tradition. Nīlakaṇṭha first states the general result[22]

$$a\left[\left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)^2+\left(\frac{1}{r}\right)^3+\ldots\right]=\frac{a}{r-1}\ .$$

where the left hand side is an infinite geometric series with the successive terms being obtained by dividing by a *cheda* (common divisor), r, assumed to be greater than 1. Nīlakaṇṭha notes that this result is best

[22] *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha, *Gaṇitapāda*, K.Sambasiva Sastri (ed.), Trivandrum 1931, p.142-143.

demonstrated by considering a particular case, say $r = 4$. Thus, what is to be demonstrated is that

$$\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \ldots = \frac{1}{3}\,.$$

Nīlakaṇṭha first obtains the sequence of results

$$\begin{aligned} \frac{1}{3} &= \frac{1}{4} + \frac{1}{(4.3)}\,, \\ \frac{1}{(4.3)} &= \frac{1}{(4.4)} + \frac{1}{(4.4.3)}\,, \\ \frac{1}{(4.4.3)} &= \frac{1}{(4.4.4)} + \frac{1}{(4.4.4.3)}\,, \end{aligned}$$

and so on, from which he derives the general result

$$\frac{1}{3} - \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n\right] = \left(\frac{1}{4}\right)^n \left(\frac{1}{3}\right).$$

Nīlakaṇṭha then goes on to present the following crucial argument to derive the sum of the infinite geometric series: As we sum more terms, the difference between $\frac{1}{3}$ and sum of powers of $\frac{1}{4}$ (as given by the right hand side of the above equation), becomes extremely small, but never zero. Only when we take all the terms of the infinite series together do we obtain the equality

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n + \ldots = \frac{1}{3}\,.$$

2.7 *Yuktibhāṣā* proofs of infinite series for π and the trigonometric functions

One of the most celebrated works in Indian mathematics and astronomy, which is especially devoted to the exposition of *yukti* or proofs, is the Malayalam work *Yuktibhāṣā* (c.1530) of Jyeṣṭhadeva.[23] Jyeṣṭhadeva

[23] *Yuktibhāṣā* of Jyeṣṭhadeva, K. Chandrasekharan (ed.), Madras 1953. *Gaṇitādhyāya* alone was edited along with notes in Malayalam by Ramavarma Thampuran and A.R.Akhileswara Aiyer, Trichur 1948. The entire work has been edited, along with an ancient Sanskrit version, *Gaṇitayuktibhāṣā* and English translation, by K.V.Sarma, with explanatory mathematical notes by K.Ramasubramanian, M.D.Srinivas and M.S.Sriram (in press).

states that his work closely follows the renowned astronomical work *Tantrasaṅgraha* (c.1500) of Nīlakaṇṭha Somasutvan and is intended to give a detailed exposition of all the mathematics required thereof. The first half of *Yuktibhāṣā* deals with various mathematical topics in seven chapters and the second half deals with all aspects of mathematical astronomy in eight chapters. The mathematical part includes a detailed exposition of proofs for the infinite series and fast converging approximations for π and the trigonometric functions, which were discovered by Mādhava (c.1375). We present an outline of some of these proofs in Appendix B.

3 *Upapatti* and "Proof"

3.1 Mathematics as a search for infallible eternal truths

The notion of *upapatti* is significantly different from the notion of 'proof' as understood in the Greek as well as the modern Western tradition of mathematics. The ideal of mathematics in the Greek and modern Western traditions is that of a formal axiomatic deductive system; it is believed that mathematics is and ought to be presented as a set of formal derivations from formally stated axioms. This ideal of mathematics is intimately linked with another philosophical presupposition – that mathematics constitutes a body of infallible eternal truths. Perhaps it is only the ideal of a formal axiomatic deductive system that could presumably measure up to this other ideal of mathematics being a body of infallible eternal truths. It is this quest for securing certainty of mathematical knowledge, which has motivated most of the foundational and philosophical investigations into mathematics and shaped the course of mathematics in the Western tradition, from the Greeks to the contemporary times.

The Greek view of mathematical objects and the nature of mathematical knowledge is clearly set forth in the following statement of Proclus (c. 5^{th} century AD) in his famous commentary on the Elements of Euclid:

> Mathematical being necessarily belongs neither among the first nor among the last and least simple kinds of being, but occupies the middle ground between the partless realities –

> simple, incomposite and indivisible – and divisible things characterized by every variety of composition and differentiation. The unchangeable, stable and incontrovertible character of the propositions about it shows that it is superior to the kind of things that move about in matter ...
>
> It is for this reason, I think, that Plato assigned different types of knowing to the highest, the intermediate, and the lowest grades of reality. To indivisible realities he assigned intellect, which discerns what is intelligible with simplicity and immediacy, and by its freedom from matter, its purity, and its uniform mode of coming in contact with being is superior to all other forms of knowledge. To divisible things in the lowest level of nature, that is, to all objects of sense perception, he assigned opinion, which lays hold of truth obscurely, whereas to intermediates, such as the forms studied by mathematics, which fall short of indivisible but are superior to divisible nature, he assigned understanding....
>
> Hence Socrates describes the knowledge of understandables as being more obscure than the highest science but clearer than the judgements of opinion. For, the mathematical sciences are more explicative and discursive than intellectual insight but are superior to opinion in the stability and irrefutability of their ideas. And their proceeding from hypothesis makes them inferior to highest knowledge, while their occupation with immaterial objects makes their knowledge more perfect than sense perception.[24]

While the above statement of Proclus is from the Platonist school, the Aristotelean tradition also held more or less similar views on the nature of mathematical knowledge, as may be seen from the following extract from the canonical text on Mathematical Astronomy, the Almagest of Claudius Ptolemy (c.2^{nd} century AD):

> For Aristotle divides theoretical philosophy too,very fittingly, into three primary categories, physics, mathematics and theology. For everything that exists is composed of matter, form

[24]Proclus: *A Commentary on the First Book of Euclid's Elements*, Tr.G.R.Morrow, Princeton, 1970, p.3,10.

and motion; none of these [three] can be observed in its substratum by itself, without the others: they can only be imagined. Now the first cause of the first motion of the universe, if one considers it simply, can be thought of as invisible and motionless deity; the division [of theoretical philosophy] concerned with investigating this [can be called] 'theology', since this kind of activity, somewhere up in the highest reaches of the universe, can only be imagined, and is completely separated from perceptible reality. The division [of theoretical philosophy] which investigates material and ever-moving nature, and which concerns itself with 'white', 'hot', 'sweet', 'soft' and suchlike qualities one may call 'physics'; such an order of being is situated (for the most part) amongst corruptible bodies and below the lunar sphere. That division [of theoretical philosophy] which determines the nature involved in forms and motion from place to place, and which serves to investigate shape, number, size and place, time and suchlike, one may define as 'mathematics'. Its subject-matter falls as it were in the middle between the other two, since, firstly, it can be conceived of both with and without the aid of the senses, and, secondly, it is an attribute of all existing things without exception, both mortal and immortal: for those things which are perpetually changing in their inseparable form, it changes with them, while for eternal things which have an aethereal nature, it keeps their unchanging form unchanged.

From all this we concluded: that the first two divisions of theoretical philosophy should rather be called guesswork than knowledge, theology because of its completely invisible and ungraspable nature, physics because of the unstable and unclear nature of matter; hence there is no hope that philosophers will ever be agreed about them; and that only mathematics can provide sure and unshakeable knowledge to its devotees, provided one approaches it rigorously. For its kind of proof proceeds by indisputable methods, namely arithmetic and geometry. Hence we are drawn to the investigation of that part of theoretical philosophy, as far as we were able to the whole of it, but especially to the theory concerning

> the divine and heavenly things. For that alone is devoted to the investigation of the eternally unchanging. For that reason it too can be eternal and unchanging (which is a proper attribute of knowledge) in its own domain, which is neither unclear nor disorderly.[25]

The view, that it is mathematics which can provide "sure and unshakeable knowledge to its devotees" has persisted in the Greco-European tradition down to the modern times. For instance, we may cite the popular mathematician philosopher of our times, Bertrand Russel, who declares, "I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere". In a similar vein, David Hilbert, one of the foremost mathematicians of our times declared, "The goal of my theory is to establish once and for all the certitude of mathematical methods".[26]

3.2 The *raison d'être* of *Upapatti*

Indian epistemological position on the nature and validation of mathematical knowledge is very different from that in the Western tradition. This is brought out for instance by the Indian understanding of what indeed is the purpose or *raison d'être* of an *upapatti*. In the beginning of the *golādhyāya* of *Siddhāntaśiromaṇi*, Bhāskarācārya says:

> मध्याद्यं द्युसदां यदत्र गणितं तस्योपपत्तिं विना
> प्रौढिं प्रौढसभासु नैति गणको निःसंशयो न स्वयम् ।
> गोले सा विमला करामलकवत् प्रत्यक्षतो दृश्यते
> तस्मादस्म्युपपत्तिबोधविधये गोलप्रबन्धोद्यतः ॥[27]

> Without the knowledge of *upapatti*-s, by merely mastering the *gaṇita* (calculational procedures) described here, from

[25] *The Almagest* of Ptolemy, Translated by G.J.Toomer, London 1984, p.36-7.

[26] Both quotations cited in Ruben Hersh, 'Some Proposals for Reviving the Philosophy of Mathematics', Adv. Math. **31**, 31-50, 1979.

[27] *Siddhāntaśiromaṇi* of Bhāskarācārya with *Vāsanābhāṣya* and *Vāsanāvārttika* of Nṛsiṃha Daivajña, Muralidhara Chaturveda (ed.), Varanasi 1981, p.326.

> the *madhyamādhikara* (the first chapter of *Siddhāntaśiromaṇi*) onwards, of the (motion of the) heavenly bodies, a mathematician will not have any value in the scholarly assemblies; without the *upapatti*-s he himself will not be free of doubt (*niḥsaṃśaya*). Since *upapatti* is clearly perceivable in the (armillary) sphere like a berry in the hand, I therefore begin the *golādhyāya* (section on spherics) to explain the *upapatti*-s.

As the commentator Nṛsiṃha Daivajña explains, 'the *phala* (object) of *upapatti* is *pāṇḍitya* (scholarship) and also removal of doubts (for oneself) which would enable one to reject wrong interpretations made by others due to *bhrānti* (confusion) or otherwise.'[28]

The same view is reiterated by Gaṇeśa Daivajña in his preface to *Buddhivilāsinī*:

> व्यक्ते वाव्यक्तसंज्ञे यदुदितमखिलं नोपपत्तिं विना तत्
> निर्भ्रान्तो वा ऋते तां सुगणकसदसि प्रौढतां नैति चायम् ।
> प्रत्यक्षं दृश्यते सा करतलकलितादर्शवत् सुप्रसन्ना
> तस्मादग्र्योपपत्तिं निगदितुमखिलम् उत्सहे बुद्धिवृद्धौ ॥[29]

> Whatever is stated in the *vyakta* or *avyakta* branches of mathematics, without *upapatti*, will not be rendered *nirbhrānta* (free from confusion); will not have any value in an assembly of mathematicians. The *upapatti* is directly perceivable like a mirror in hand. It is therefore, as also for the elevation of the intellect (*buddhi-vṛddhi*), that I proceed to enunciate *upapatti*-s in entirety.

Thus as per the Indian mathematical tradition, the purpose of *upapatti* is mainly (i) To remove doubts and confusion regarding the validity and interpretation of mathematical results and procedures; and, (ii) To obtain assent in the community of mathematicians.

Further, in the Indian tradition, mathematical knowledge is not taken to be different in any 'fundamental sense' from that in natural sci-

[28] *Siddhantaśiromaṇi* , cited above, p.326.
[29] *Buddhivilāsinī*, cited above, p.3.

ences. The valid means for acquiring knowledge in mathematics are the same as in other sciences: *Pratyakṣa* (perception), *Anumāna* (inference), *Śabda* or *Āgama* (authentic tradition). In his *Vāsanābhāṣya* on *Siddhāntaśiromaṇi* Bhāskarācārya refers to the sources of valid knowledge (*pramāṇa*) in mathematical astronomy, and declares that

> यद्येवमुच्यते गणितस्कन्धे उपपत्तिमान् एवागमः प्रमाणम् ।[30]
>
> For all that is discussed in Mathematical Astronomy, only an authentic tradition or established text which is supported by *upapatti* will be a *pramāṇa*.

Upapatti here includes observation. Bhāskarācārya, for instance, says that the *upapatti* for the mean periods of planets involves observations over very long periods.

3.3 The limitations of *Tarka* or proof by contradiction

An important feature that distinguishes the *upapatti*-s of Indian mathematicians is that they do not generally employ the method of proof by contradiction or *reductio ad absurdum*. Sometimes arguments, which are somewhat similar to the proof by contradiction, are employed to show the non-existence of an entity, as may be seen from the following *upapatti* given by Kṛṣṇa Daivajña to show that "a negative number has no square root":

> The square-root can be obtained only for a square. A negative number is not a square. Hence how can we consider its square-root? It might however be argued: 'Why will a negative number not be a square? Surely it is not a royal fiat'... Agreed. Let it be stated by you who claim that a negative number is a square as to whose square it is: Surely not of a positive number, for the square of a positive number is always positive by the rule...Not also of a negative number. Because then also the square will be positive by the rule...This being the case, we do not see any such number whose square becomes negative...[31]

[30] *Siddhāntaśiromaṇi*, cited above, p.30.
[31] *Bījapallava*, cited earlier, p.19.

Such arguments, known as *tarka* in Indian logic, are employed only to prove the non-existence of certain entities, but not for proving the existence of an entity, which existence is not demonstrable (at least in principle) by other direct means of verification.

In rejecting the method of indirect proof as a valid means for establishing existence of an entity which existence cannot even in principle be established through any direct means of proof, the Indian mathematicians may be seen as adopting what is nowadays referred to as the 'constructivist' approach to the issue of mathematical existence. But the Indian philosophers, logicians, etc., do much more than merely disallow certain existence proofs. The general Indian philosophical position is one of eliminating from logical discourse all reference to such *aprasiddha* entities, whose existence is not even in principle accessible to all means of verification.[32] This appears to be also the position adopted by the Indian mathematicians. It is for this reason that many an "existence theorem" (where all that is proved is that the non-existence of a hypothetical entity is incompatible with the accepted set of postulates) of Greek or modern Western mathematics would not be considered significant or even meaningful by Indian mathematicians.

3.4 *Upapatti* and "Proof"

We now summarize our discussion on the classical Indian understanding of the nature and validation of mathematical knowledge:

1. The Indian mathematicians are clear that results in mathematics, even those enunciated in authoritative texts, cannot be accepted as valid unless they are supported by *yukti* or *upapatti* . It is not enough that one has merely observed the validity of a result in a large number of instances.

2. Several commentaries written on major texts of Indian mathematics and astronomy present *upapatti*-s for the results and procedures enunciated in the text.

[32] For the approach adopted by Indian philosophers to *tarka* or the method of indirect proof see for instance, M.D.Srinivas, "The Indian Approach to Formal Logic and the Methodology of Theory Construction: A Preliminary View", PPST Bulletin **9**, 32-59, 1986.

3. The *upapatti*-s are presented in a sequence proceeding systematically from known or established results to finally arrive at the result to be established.

4. In the Indian mathematical tradition the *upapatti*-s mainly serve to remove doubts and obtain consent for the result among the community of mathematicians.

5. The *upapatti*-s may involve observation or experimentation. They also depend on the prevailing understanding of the nature of the mathematical objects involved.

6. The method of *tarka* or "proof by contradiction" is used occasionally. But there are no *upapatti*-s which purport to establish existence of any mathematical object merely on the basis of *tarka* alone.

7. The Indian mathematical tradition did not subscribe to the ideal that *upapatti*-s should seek to provide irrefutable demonstrations establishing the absolute truth of mathematical results. There was apparently no attempt to present the *upapatti*-s as a part of a deductive axiomatic system. While Indian mathematics made great strides in the invention and manipulation of symbols in representing mathematical results and in facilitating mathematical processes, there was no attempt at formalization of mathematics.

The classical Indian understanding of the nature and validation of mathematical knowledge seems to be rooted in the larger epistemological perspective developed by the *Nyāya* school of Indian logic. Some of the distinguishing features of *Nyāya* logic, which are particularly relevant in this context, are: That it is a logic of cognitions (*jñāna*) and not "propositions", that it has no concept of pure "formal validity" as distinguished from "material truth", that it does not distinguish necessary and contingent truth or analytical and synthetic truth, that it does not admit, in logical discourse, premises which are known to be false or terms that are non-instantiated, that it does not accord *tarka* or "proof by contradiction" a status of independent *pramāṇa* or means of knowledge, and so on.[33]

[33] For a discussion of some of these features, see J.N.Mohanty: *Reason and Tradition in Indian Thought*, Oxford, 1992.

The close relation between the methodology of Indian mathematics and *Nyāya* epistemology, has been commented upon by a leading scholar of *navya-nyāya*:

> The western concept of proof owes its origin to Plato's distinction between knowledge and opinion or between reason and sense. According to Plato, reason not merely knows objects having ontological reality, but also yields a knowledge which is logically superior to opinion to which the senses can aspire. On this distinction is based the distinction between contingent and necessary truths, between material truth and formal truth, between rational knowledge which can be proved and empirical knowledge which can only be verified ...
>
> As a matter of fact, the very concept of reason is unknown in Indian philosophy. In the systems which accept inference as a source of true knowledge, the difference between perception and inference is not explained by referring the two to two different faculties of the subject, sense and reason, but by showing that inferential knowledge is caused in a special way by another type of knowledge (*vyāpti-jñāna* [knowledge of invariable concomitance]), whereas perception is not so caused ...
>
> In Indian mathematics we never find a list of self-evident propositions which are regarded as the basic premises from which other truths of mathematics follow ...
>
> Euclid was guided in his axiomatization of geometry by the Aristotelean concept of science as a systematic study with a few axioms which are self-evident truths. The very concept of a system thus involves a distinction between truths which need not be proved (either because they are self-evident as Aristotle thought, or because they have been just chosen as the primitive propositions of a system as the modern logicians think) and truths which require proof. But this is not enough. What is important is to suppose that the number of self-evident truths or primitive propositions is very small and can be exhaustively enumerated.

> Now there is no Indian philosophy which holds that some truths do not require any proof while others do. The systems which accept *svataḥprāmāṇyavāda* hold that all (true) knowledge is self-evidently true, and those which accept *parataḥprāmāṇyavāda* hold that all (true) knowledge requires proof; there is no system which holds that some truths require proof while others do not ...[34]

3.5 Towards a new epistemology for Mathematics

Mathematics today, rooted as it is in the modern Western tradition, suffers from serious limitations. Firstly, there is the problem of 'foundations' posed by the ideal view of mathematical knowledge as a set of infallible eternal truths. The efforts of mathematicians and philosophers of the West to secure for mathematics the status of indubitable knowledge has not succeeded; and there is a growing feeling that this goal may turn out to be a mirage.

After surveying the changing status of mathematical truth from the Platonic position of "truth in itself", through the early twentieth century position that "mathematical truth resides ... uniquely in the logical deductions starting from premises arbitrarily set by axioms", to the twentieth century developments which question the infallibility of these logical deductions themselves, Bourbaki are forced to conclude that:

> To sum up, we believe that mathematics is destined to survive, and that the essential parts of this majestic edifice will never collapse as a result of the sudden appearance of a contradiction; but we cannot pretend that this opinion rests on anything more than experience. Some will say that this is small comfort; but already for two thousand five hundred years mathematicians have been correcting their errors to the consequent enrichment and not impoverishment of this science; and this gives them the right to face the future with serenity.[35]

[34] Sibajiban Bhattacharya, 'The Concept of Proof in Indian Mathematics and Logic', in *Doubt, Belief and Knowledge*, Delhi, 1987, p.193, 196.

[35] N.Bourbaki, *Elements of Mathematics: Theory of Sets*, Springer 1968, p.13; see also N.Bourbaki, *Elements of History of Mathematics*, Springer 1994, p.1-45.

Apart from the problems inherent in the goals set for mathematics, there are also other serious inadequacies in the Western epistemology and philosophy of mathematics. The ideal view of mathematics as a formal deductive system gives rise to serious distortions. Some scholars have argued that this view of mathematics has rendered philosophy of mathematics barren and incapable of providing any understanding of the actual history of mathematics, the logic of mathematical discovery and, in fact, the whole of creative mathematical activity.[36]

There is also the inevitable chasm between the ideal notion of infallible mathematical proof and the actual proofs that one encounters in standard mathematical practice, as portrayed in a recent book:

> On the one side, we have real mathematics, with proofs, which are established by the 'consensus of the qualified'. A real proof is not checkable by a machine, or even by any mathematician not privy to the *gestalt*, the mode of thought of the particular field of mathematics in which the proof is located. Even to the 'qualified reader' there are normally differences of opinion as to whether a real proof (i.e., one that is actually spoken or written down) is complete or correct. These doubts are resolved by communication and explanation, never by transcribing the proof into first order predicate calculus. Once a proof is 'accepted', the results of the proof are regarded as true (with very high probability). It may take generations to detect an error in a proof... On the other side, to be distinguished from real mathematics, we have 'meta-mathematics'... It portrays a structure of proofs, which are indeed infallible 'in principle'... [The philosophers of mathematics seem to claim] that the problem of fallibility in real proofs... has been conclusively settled by the presence of a notion of infallible proof in meta-mathematics... One wonders how they would justify such a claim.[37]

Apart from the fact that the modern Western epistemology of mathematics fails to give an adequate account of the history of mathematics

[36] I.Lakatos, *Proofs and Refutations: The Logic of Mathematical Discovery*, Cambridge 1976.

[37] Philip J.Davis and Reuben Hersh, *The Mathematical Experience*, Boston, 1981, p.354-5.

and standard mathematical practice, there is also the growing awareness that the ideal of mathematics as a formal deductive system has had serious consequences in the teaching of mathematics. The formal deductive format adopted in mathematics books and articles greatly hampers understanding and leaves the student with no clear idea of what is being talked about.

Notwithstanding all these critiques, it is not likely that, within the Western philosophical tradition, any radically different epistemology of mathematics will emerge; and so the driving force for modern mathematics is likely to continue to be a search for infallible eternal truths and modes of establishing them, in one form or the other. This could lead to 'progress' in mathematics, but it would be progress of a rather limited kind.

If there is a major lesson to be learnt from the historical development of mathematics, it is perhaps that the development of mathematics in the Greco-European tradition was seriously impeded by its adherence to the cannon of ideal mathematics as laid down by the Greeks. In fact, it is now clearly recognized that the development of mathematical analysis in the Western tradition became possible only when this ideal was given up during the heydays of the development of "infinitesimal calculus" during $16^{th} - 18^{th}$ centuries. As one historian of mathematics notes:

> It is somewhat paradoxical that this principal shortcoming of Greek mathematics stemmed directly from its principal virtue–the insistence on absolute logical rigour...Although the Greek bequest of deductive rigour is the distinguishing feature of modern mathematics, it is arguable that, had all the succeeding generations also refused to use real numbers and limits until they fully understood them, the calculus might never have been developed and mathematics might now be a dead and forgotten science.[38]

It is of course true that the Greek ideal has gotten reinstated at the heart of mathematics during the last two centuries, but it seems that most of the foundational problems of mathematics can also be perhaps traced to the same development. In this context, study of alternative epistemologies such as that developed in the Indian tradition of mathematics, could prove to be of great significance for the future of mathematics.

[38] C.H.Edwards, *History of Calculus*, New York 1979, p.79.

Appendices

A List of Works Containing *Upapatti*-s

The following are some of the important commentaries available in print, which present *upapatti*-s of results and procedures in mathematics and astronomy:

1. *Bhāṣya* of Bhāskara I (c.629) on *Āryabhaṭīya* of Āryabhaṭa (c.499), K.S.Shukla (ed.), New Delhi 1975.

2. *Bhāṣya* of Govindasvāmin (c.800) on *Mahābhāskarīya* of Bhāskara I (c.629), T.S.Kuppanna Sastri (ed.), Madras 1957.

3. *Vāsanābhāṣya* of Caturveda Pṛthūdakasvāmin (c.860) on *Brāhmasphuṭasiddhānta* of Brahmagupta (c.628), Chs. I-III, XXI, Ramaswarup Sharma (ed.), New Delhi 1966; Ch XXI, Edited and Translated by Setsuro Ikeyama, Ind. Jour. Hist. Sc. Vol. **38**, 2003.

4. *Vivaraṇa* of Bhāskarācārya II (c.1150) on *Śiṣyadhīvṛddhidatantra* of Lalla (c.748), Chandrabhanu Pandey (ed.), Varanasi 1981.

5. *Vāsanā* of Bhāskarācārya II (c.1150) on his own *Bījagaṇita*, Jivananda Vidyasagara (ed.), Calcutta 1878; Achyutananda Jha (ed.) Varanasi 1949, Rep. 1994.

6. *Mitākṣarā* or *Vāsanā* of Bhāskarācārya II (c.1150) on his own *Siddhāntaśiromaṇi*, Bapudeva Sastrin (ed.), Varanasi 1866; Muralidhara Chaturveda (ed.), Varanasi 1981.

7. *Vāsanābhāṣya* of Āmarāja (c.1200) on *Khaṇḍakhādyaka* of Brahmagupta (c.665), Babuaji Misra (ed.), Calcutta 1925.

8. *Gaṇitabhūṣaṇa* of Makkībhaṭṭa (c.1377) on *Siddhāntaśekhara* of Śrīpati (c.1039), Chs. I-III, Babuaji Misra (ed.), Calcutta 1932.

9. *Siddhāntadīpikā* of Parameśvara (c.1431) on the *Bhāṣya* of Govindasvāmin (c.800) on *Mahābhāskarīya* of Bhāskara I (c.629), T.S. Kuppanna Sastri (ed.), Madras 1957.

10. *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha Somasutvan (c.1501) on *Āryabhaṭīya* of Āryabhaṭa (c.499), K. Sambasiva Sastri (ed.), 3 Vols., Trivandrum 1931, 1932, 1957.

11. *Yuktibhāṣā* (in Malayalam) of Jyeṣṭhadeva (c.1530); *Gaṇitādhyāya*, RamaVarma Thampuran and A.R. Akhileswara Aiyer (eds.), Trichur 1948; K.Chandrasekharan (ed.), Madras 1953. Edited and Translated by K.V. Sarma with Explanatory Notes by K. Ramasubramanian, M.D. Srinivas and M.S. Sriram (in Press).

12. *Yuktidīpikā* of Śaṅkara Vāriyar (c.1530) on *Tantrasaṅgraha* of Nīlakaṇṭha Somasutvan (c.1500), K.V.Sarma (ed.), Hoshiarpur 1977.

13. *Kriyākramakarī* of Śaṅkara Vāriyar (c.1535) on *Līlāvatī* of Bhāskarācārya II (c.1150), K.V.Sarma (ed.), Hoshiarpur 1975.

14. *Sūryaprakāśa* of Sūryadāsa (c.1538) on Bhāskarācārya's *Bījagaṇita* (c.1150), Chs. I-V, Edited and translated by Pushpa Kumari Jain, Vadodara 2001.

15. *Buddhivilāsinī* of Gaṇeśa Daivajña (c.1545) on *Līlāvatī* of Bhāskarācārya II (c.1150), V.G.Apte (ed.), 2 Vols, Pune 1937.

16. *Ṭīkā* of Mallāri (c.1550) on *Grahalāghava* of Gaṇeśa Daivajña (c.1520), Balachandra (ed.), Varanasi 1865; Kedaradatta Joshi (ed.), Varanasi 1981.

17. *Bījanavāṅkurā* or *Bījapallavam* of Kṛṣṇa Daivajña (c.1600) on *Bījagaṇita* of Bhāskarācārya II (c.1150), V.G.Apte (ed.), Pune 1930; T.V.Radha Krishna Sastri (ed.), Tanjore 1958; Biharilal Vasistha (ed.), Jammu 1982.

18. *Śiromaṇiprakāśa* of Gaṇeśa (c.1600) on *Siddhāntaśiromaṇi* of Bhāskarācārya II (c.150), *Grahagaṇitādhyāya*, V.G.Apte (ed.), 2 Vols. Pune 1939, 1941.

19. *Gūḍhārthaprakāśa* of Raṅganātha (c.1603) on *Sūryasiddhānta*, Jivananda Vidyasagara (ed.), Calcutta 1891; Reprint, Varanasi 1990.

20. *Vāsanāvārttika*, commentary of Nṛsiṃha Daivajña (c.1621) on *Vāsanābhāṣya* of Bhāskarācārya II, on his own *Siddhāntaśiromaṇi* (c.1150), Muralidhara Chaturveda (ed.), Varanasi 1981.

21. *Marīci* of Munīśvara (c.1630) on *Siddhantaśiromaṇi* of Bhāskarācārya (c.1150), *Madhyamādhikāra*, Muralidhara Jha (ed.), Varanasi 1908; *Grahagaṇitādhyāya*, Kedaradatta Joshi (ed.), 2 vols. Varanasi 1964; *Golādhyāya*, Kedaradatta Joshi (ed.), Delhi 1988.

22. *Āśayaprakāśa* of Munīśvara (c.1646) on his own *Siddhāntasārvabhauma*, *Gaṇitādhyāya* Chs. I-II, Muralidhara Thakura (ed.), 2 Vols, Varanasi 1932, 1935; Chs. III-IX, Mithalal Ojha (ed.), Varanasi 1978.

23. *Śeṣavāsanā* of Kamalākarabhaṭṭa (c.1658) on his own *Siddhāntatattvaviveka*, Sudhakara Dvivedi (ed.), Varanasi 1885; Reprint, Varanasi 1991.

24. *Sauravāsanā* of Kamalākarabhaṭṭa (c.1658) on *Sūryasiddhānta*, Chs. I-X, Sri Chandra Pandeya (ed.), Varanasi 1991.

25. *Gaṇitayuktayaḥ*, Tracts on Rationale in Mathematical Astronomy by Various Kerala Astronomers (c.$16^{th} - 19^{th}$ century), K.V.Sarma (ed.), Hoshiarpur 1979.

B Some *Upapatti*-s from *Yuktibhāṣā* (c.1530)

In this Appendix we shall present some of the proofs contained in the Mathematics part of the celebrated Malayalam text *Yuktibhāṣā* [39] of Jyeṣṭhadeva (c.1530). This part is divided into seven chapters, of which the last two, entitled *Paridhi* and *Vyāsa* (Circumference and Diameter) and *Jyānayana* (Computation of Sines), contain many important results concerning infinite series and fast convergent approximations

[39] *Yuktibhāṣā* (in Malayalam) of Jyeṣṭhadeva (c.1530); *Gaṇitādhyāya*, Ramavarma Thampuran and A.R. Akhileswara Aiyer (eds.), Trichur 1947; K. Chandrasekharan (ed.), Madras 1953; Edited, along with an ancient Sanskrit version *Gaṇitayuktibhāṣā* and English Translation, by K.V.Sarma, with Explanatory Notes by K.Ramasubramanian, M.D.Srinivas and M.S.Sriram (in press).

for π and the trigonometric functions. In the preamble to his work, Jyeṣṭhadeva states that his work gives an exposition of the mathematics necessary for the computation of planetary motions as expounded in *Tantrasaṅgraha* of Nīlakaṇṭha (c.1500). The proofs given in *Yuktibhāṣā* have been reproduced (mostly in the form of Sanskrit verses or *kārikās*) by Śaṅkara Vāriyar in his commentaries *Yuktidīpikā* [40] on *Tantrasaṅgraha* and *Kriyākramakarī* [41] on *Līlāvatī*. Since the later work is considered to be written around 1535 A.D., the time of composition of *Yuktibhāṣā* may reasonably be placed around 1530 A.D.

In what follows, we shall present a brief outline of some of the mathematical topics and proofs given in Chapters VI and VII of *Yuktibhāṣā*, following closely the order which they appear in the text.

B.1 Chapter VI : *Paridhi and Vyāsa* (Circumferene and Diameter)

The chapter starts with a proof of *bhujā-koṭi-karṇa-nyāya* (the so called Pythagoras theorem), which has also been proved earlier in the first chapter of the work.[42] It is then followed by a discussion of how to arrive at successive approximations to the circumference of a circle by giving a systematic procedure for computing successively the perimeters of circumscribing square, octagon, regular polygon of sides 16, 32, and so on. The treatment of infinite series expansions is taken up thereafter.

B.1.1 To obtain the circumference without calculating square-roots

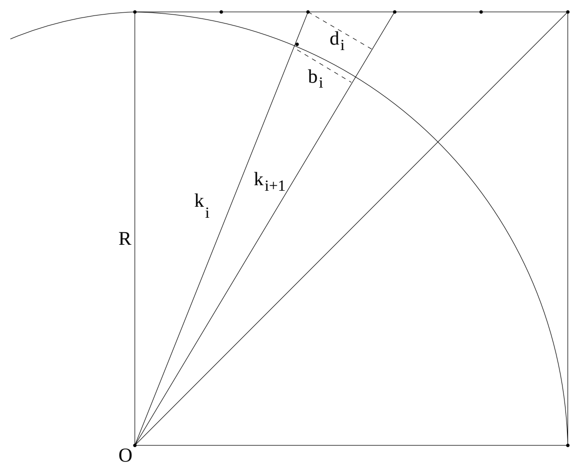
Consider a quadrant of the circle, inscribed in a square and divide a side of the square, which is tangent to the circle, into a large number of equal

[40] *Yuktidīpikā* of Sankara Variyar (c.1530) on *Tantrasaṅgraha* of Nīlakaṇṭha Somasutvan (c.1500), K.V.Sarma (ed.), Hoshiarpur 1977. At the end of each chapter of this work, Śaṅkara states that he is only presenting the material which has been well expounded by the great *dvija* of the Parakroḍha house, Jyeṣṭhadeva.

[41] *Kriyākramakarī* of Śaṅkara Vāriyar (c.1535) on *Līlāvatī* of Bhāskarācārya II (c.1150), K.V.Sarma (ed.), Hoshiarpur 1975.

[42] In fact, according to *Yuktibhāṣā* , almost all mathematical computations are pervaded (*vyāpta*) by the *trairāśika-nyāya* (the rule of proportion as exemplified for instance in the case of similar triangles)and the *bhujā-koṭi-karṇa-nyāya*.

parts. The more the number of divisions the better is the approximation to the circumference.



$C/8$ (one eighth of the circumference) is approximated by the sum of the *jyārdhās* (half-chords) b_i of the arc-bits to which the circle is divided by the *karṇas* (hypotenuses) which join the points which the divide tangent to the centre of the circle. Let k_i be the length of the i^{th} *karṇa*. Then,

$$b_i = \left(\frac{R}{k_i}\right) d_i = \frac{R}{k_i}\left[\left(\frac{R}{n}\right)\frac{R}{k_{i+1}}\right] = \left(\frac{R}{n}\right)\frac{R^2}{k_i k_{i+1}}$$

Hence

$$\frac{\pi}{4} = \frac{C}{8R} = \left(\frac{1}{n}\right)\sum_{i=0}^{n-1}\frac{R^2}{k_i k_{i+1}} \approx \left(\frac{1}{n}\right)\sum_{i=0}^{n-1}\left(\frac{R^2}{k_i}\right)^2$$
$$= \left(\frac{1}{n}\right)\sum_{i=0}^{n-1}\frac{R^2}{\left[R^2 + i^2\left(\frac{R}{n}\right)^2\right]}$$

Series expansion of each term in the RHS is obtained by iterating the relation

$$\frac{a}{b} = \frac{a}{c} - \left(\frac{a}{b}\right)\left(\frac{b-c}{c}\right),$$

which leads to

$$\frac{a}{b} = \frac{a}{c} - \left(\frac{a}{b}\right)\left(\frac{b-c}{c}\right) + \left(\frac{a}{c}\right)\left(\frac{b-c}{c}\right)^2 + \ldots.$$

This (binomial) expansion is also justified later by showing how the partial sums in the following series converge to the result.

$$\frac{100}{10} = \frac{100}{8} - \left(\frac{100}{10}\right)\left(\frac{10-8}{8}\right) + \left(\frac{100}{8}\right)\left(\frac{10-8}{8}\right)^2 - \ldots.$$

Thus

$$\frac{\pi}{4} = 1 - \left(\frac{1}{n}\right)^3 \sum_{i=1}^{n} i^2 + \left(\frac{1}{n}\right)^5 \sum_{i=1}^{n} i^4 - \ldots$$

When n becomes very large, this leads to the series given in the rule of Mādhava *vyāse vāridhinihate* ...[43]

$$\frac{C}{4D} = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$$

B.1.2 *Samaghāta-saṅkalita* – Sums of powers of natural numbers

In the above derivation, the following estimate was employed for the *samaghāta-saṅkalita* of order k, for large n :

$$S_n^{(k)} = 1^k + 2^k + 3^k + \ldots + n^k \approx \frac{n^{k+1}}{(k+1)}$$

This is proved first for the case of *mūla-saṅkalita*

$$\begin{aligned} S_n^{(1)} &= 1 + 2 + 3 + \ldots + n \\ &= [n - (n-1)] + [n - (n-2)] + \cdots + n \\ &= n^2 - S_{n-1}^{(1)} \end{aligned}$$

Hence for large n,

$$S_n^{(1)} \approx \frac{n^2}{2}$$

Then, for the *varga-saṅkalita* and the *ghana-saṅkalita*, the following estimates are proved for large n:

$$\begin{aligned} S_n^{(2)} &= 1^2 + 2^2 + 3^2 + \ldots + n^2 \approx \frac{n^3}{3} \\ S_n^{(3)} &= 1^3 + 2^3 + 3^3 + \ldots + n^3 \approx \frac{n^4}{4} \end{aligned}$$

[43] This result is attributed to Mādhava by Śaṅkara Vāriyar in *Kriyākramakarī*, cited earlier, p.379; see also *Yuktidīpikā* , cited earlier, p.101.

It is then observed that, in each case, the derivation above is based on the result

$$nS_n^{(k-1)} - S_n^{(k)} = S_{n-1}^{(k-1)} + S_{n-2}^{(k-1)} + \ldots + S_1^{(k-1)}$$

It is observed that the right hand side of the above equation is a repeated sum of the lower order $(k-1)$ *saṅkalita*. Now if we have already estimated this lower order *saṅkalita*, $S_n^{(k-1)} \approx \frac{n^k}{k}$, then

$$\begin{aligned} nS_n^{(k-1)} - S_n^{(k)} &\approx \frac{(n-1)^k}{k} + \frac{(n-2)^k}{k} + \frac{(n-3)^k}{k} + \ldots \\ &\approx \left(\frac{1}{k}\right) S_n^{(k)}. \end{aligned}$$

Hence, for the general *samaghāta-saṅkalita*, we obtain the estimate

$$S_n^{(k)} \approx \frac{n^{k+1}}{(k+1)}.$$

B.1.3 *Vāra-saṅkalita* – Repeated summations

The *vāra-saṅkalita* or *saṅkalita-saṅkalita* or repeated sums, are defined as follows:

$$\begin{aligned} V_n^{(1)} &= S_n^{(1)} = 1 + 2 + \ldots + (n-1) + n \\ V_n^{(r)} &= V_1^{(r-1)} + V_2^{(r-1)} + \ldots + V_n^{(r-1)} \end{aligned}$$

It is shown that, for large n

$$V_n^{(r)} \approx \frac{n^{r+1}}{(r+1)!}.$$

B.1.4 *Cāpīkaraṇa* – Determination of the arc

This can be done by the series given by the rule[44] *iṣṭajyātrijyayorghātāt* ... which is derived in the same way as the above series for $\frac{C}{8}$.

$$R\theta = R\left(\frac{\sin\theta}{\cos\theta}\right) - \frac{R}{3}\left(\frac{\sin\theta}{\cos\theta}\right)^3 + \frac{R}{5}\left(\frac{\sin\theta}{\cos\theta}\right)^5 - \ldots$$

[44] See for instance, *Kriyākramakarī*, cited earlier, p.95-96.

It is noted that $|\frac{\sin\theta}{\cos\theta}| \leq 1$, is a necessary condition for the terms in the above series to progressively lead to the result. Using the above, for $\theta = \frac{\pi}{6}$, the following series is obtained:

$$C = \sqrt{12D^2}\left[1 - \frac{1}{(3.3)} + \frac{1}{(3^2.5)} - \frac{1}{(3^3.7)} + \ldots\right].$$

B.1.5 *Antya-saṃskāra* – Correction term to obtain accurate circumference

Let us set

$$\frac{C}{4D} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^{n-1}\frac{1}{(2n-1)} + (-1)^n\frac{1}{a_n}.$$

Then the *saṃskāra-hāraka* (correction divisor), a_n will be accurate if

$$\frac{1}{a_n} + \frac{1}{a_{n+1}} = \frac{1}{2n+1}.$$

This leads to the successive approximations:[45]

$$\begin{aligned}
\frac{\pi}{4} &\approx 1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^{n-1}\frac{1}{(2n-1)} + (-1)^n\frac{1}{4n}, \\
\frac{\pi}{4} &\approx 1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^{n-1}\frac{1}{(2n-1)} + (-1)^n\frac{1}{4n + \frac{4}{4n}}, \\
&= 1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^{n-1}\frac{1}{(2n-1)} + (-1)^n\frac{n}{(4n^2+1)}.
\end{aligned}$$

Later at the end of the chapter, the rule[46] *ante samasaṅkhyādalavargaḥ* . . . , is cited as the *sūkṣmatara-saṃskāra*, a much more accurate correction:

$$\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{(-1)^{n-1}}{(2n-1)} + \frac{(-1)^n(n^2+1)}{(4n^3+5n)},$$

[45] These are attributed to Mādhava in *Kriyākramakarī*, cited earlier, p.279; also cited in *Yuktidīpikā* , cited earlier, p.101.

[46] *Kriyākramakarī*, cited earlier, p.390; *Yuktidīpikā* , cited earlier, p.103.

B.1.6 Transformation of series

The above correction terms can be used to transform the series for the circumference as follows:

$$\frac{C}{4D} = \frac{\pi}{4} = \left[1 - \frac{1}{a_1}\right] - \left[\frac{1}{3} - \frac{1}{a_1} - \frac{1}{a_2}\right] + \left[\frac{1}{5} - \frac{1}{a_2} - \frac{1}{a_3}\right] \ldots$$

It is shown that, using the second order correction terms, we obtain the following series given by the rule[47] *samapañcāhatayoḥ ...*

$$\frac{C}{16D} = \frac{1}{(1^5 + 4.1)} - \frac{1}{(3^5 + 4.3)} + \frac{1}{(5^5 + 4.5)} - \ldots$$

It is also noted that by using merely the lowest order correction terms, we obtain the following series given by the rule[48] *vyāsād vāridhinihatāt* ...

$$\frac{C}{4D} = \frac{3}{4} + \frac{1}{(3^3 - 3)} - \frac{1}{(5^3 - 5)} + \frac{1}{(7^3 - 7)} - \ldots$$

B.1.7 Other series expansions

It is further noted, by using non-optimal correction divisors in the above transformed series, one can also obtain the following results given in the rules[49] *dvyādiyujāṃ vā kṛtayo* ... and *dvyādeścaturādervā ...*

$$\begin{aligned}
\frac{C}{4D} &= \frac{1}{2} + \frac{1}{(2^2 - 1)} - \frac{1}{(4^2 - 1)} + \frac{1}{(6^2 - 1)} - \ldots \\
\frac{C}{8D} &= \frac{1}{2} - \frac{1}{(4^2 - 1)} - \frac{1}{(8^2 - 1)} - \frac{1}{(12^2 - 1)} - \ldots \\
\frac{C}{8D} &= \frac{1}{(2^2 - 1)} + \frac{1}{(6^2 - 1)} + \frac{1}{(10^2 - 1)} - \ldots
\end{aligned}$$

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[47] *Kriyākramakarī*, cited earlier, p.390; *Yuktidīpikā* , cited earlier, p.102.
[48] *Kriyākramakarī*, cited earlier, p.390; *Yuktidīpikā* , cited earlier, p.102.
[49] *Kriyākramakarī*, cited earlier, p.390; *Yuktidīpikā* , cited earlier, p.103.

B.2 Chapter VII : *Jyānayanam* – Computation of Sines

B.2.1 *Jyā, koti* and *śara* – $R\sin x,\ R\cos x$ and $R(1-\cos x)$

First is discussed the construction of an inscribed regular hexagon with side equal to the radius, which gives the value of $R\sin(\frac{\pi}{6})$. Then are derived the relations:

$$\begin{aligned} R\sin\left(\frac{\pi}{2}-x\right) &= R\cos x = R(1-\text{versin } x) \\ R\sin\left(\frac{x}{2}\right) &= \frac{1}{2}[(R\sin x)^2+(R\text{ versin } x)^2]^{\frac{1}{2}}. \end{aligned}$$

Using the above relations several Rsines can be calculated starting from the following:

$$\begin{aligned} R\sin\left(\frac{\pi}{6}\right) &= \frac{R}{2} \\ R\sin\left(\frac{\pi}{2}\right) &= \left(\frac{R^2}{2}\right)^{\frac{1}{2}}. \end{aligned}$$

This is one way of determining the *paṭhita-jyā* (enunciated or tabulated sine values), when a quadrant of a circle is divided into 24 equal parts of $3°45' = 225'$ each. To find the Rsines of intermediate values, a first approximation is

$$R\sin(x+h) \approx R\sin x + h\, R\cos x.$$

Then is derived the following better approximation as given in the rule[50] *iṣṭadoḥkoṭidhanuṣoḥ* . . . :

$$\begin{aligned} R\sin(x+h) &\approx R\sin x + \left(\frac{2}{\Delta}\right)\left(R\cos x - \left(\frac{1}{\Delta}\right)R\sin x\right) \\ R\cos(x+h) &\approx R\cos x + \left(\frac{2}{\Delta}\right)\left(R\sin x - \left(\frac{1}{\Delta}\right)R\cos x\right), \end{aligned}$$

where $\Delta = \frac{2R}{h}$.

[50] *Tantrasaṅgraha* , 2.10-14.

B.2.2 Accurate determination of sines

Given an arc $s = Rx$, divide it into n equal parts and let the *piṇḍa-jyās* B_j, and *śaras* $S_{j-\frac{1}{2}}$, with $j = 0, 1, \ldots n$, be given by

$$\begin{aligned} B_j &= R \sin\left(\frac{jx}{n}\right), \\ S_{j-\frac{1}{2}} &= R \text{ vers } \left[\frac{(j-\frac{1}{2})x}{n}\right] \end{aligned}$$

If α be the *samasta-jyā* (total chord) of the arc $\frac{s}{n}$, then the second order sine difference (*jyā-khaṇḍāntara*) is shown to satisfy

$$\begin{aligned} (B_{j+1} - B_j) - (B_j - B_{j-1}) &= \left(\frac{\alpha}{r}\right)(S_{j-\frac{1}{2}} - S_{j+\frac{1}{2}}) \\ &= \left(\frac{\alpha}{r}\right)^2 B_j, \end{aligned}$$

for $j = 1, 2, \ldots n$. From this are derived the relations

$$\begin{aligned} S_{n-\frac{1}{2}} - S_{\frac{1}{2}} &= \left(\frac{\alpha}{r}\right)(B_1 + B_2 + \ldots + B_{n-1}), \\ B_n - nB_1 &= -\left(\frac{\alpha}{R}\right)^2 [B_1 + (B_1 + B_2) + \ldots + (B_1 + \ldots + B_{n-1})] \\ &= -\left(\frac{\alpha}{r}\right)\left[S_{\frac{1}{2}} + S_{\frac{3}{2}} + \cdots + S_{n-\frac{1}{2}} - nS_{\frac{1}{2}}\right] \end{aligned}$$

If B and S are the *jyā* and *śara* of the arc s, then it is noted that, in the limit of very large n, we have as a first approximation

$$B_n \approx B, \;\; B_j \approx \frac{js}{n}, \;\; S_{n-\frac{1}{2}} \approx S, S_{\frac{1}{2}} \approx 0 \;\text{ and }\; \alpha \approx \frac{s}{n}.$$

Hence

$$S \;\approx\; \left(\frac{1}{R}\right)\left(\frac{s}{n}\right)^2 [1 + 2 + \ldots + n - 1] \approx \frac{s^2}{2R}.$$

and

$$\begin{aligned} B &\approx s - \left(\frac{1}{R}\right)^2 \left(\frac{s}{n}\right)^3 [1 + (1+2) + \ldots + (1 + 2 + \ldots + n - 1))] \\ &\approx s - \frac{s^3}{6R^2}. \end{aligned}$$

Iterating these results we get successive approximations for the difference between the Rsine and the arc (*jyā-cāpāntara*), leading to the following series given by the rule[51] *nihatya cāpavargeṇa* ...:

$$R\sin\left(\frac{s}{R}\right) = B = R\left[\left(\frac{s}{R}\right) - \frac{\left(\frac{s}{R}\right)^3}{3!} + \frac{\left(\frac{s}{R}\right)^5}{5!} - \ldots\right]$$

$$R - R\cos\left(\frac{s}{R}\right) = S = R\left[\left(\frac{s}{R}\right)^2 - \frac{\left(\frac{s}{R}\right)^4}{4!} + \frac{\left(\frac{s}{R}\right)^6}{6!} - \ldots\right]$$

While carrying successive approximations, the following result for *vāra-saṅkalitas* (repeated summations) is used:

$$\sum_{j=1}^{n} \frac{j(j+1)\ldots(j+k-1)}{k!} = \frac{n(n+1)(n+2)\ldots(n+k)}{(k+1)!} \approx \frac{n^{k+1}}{(k+1)!}.$$

Then is obtained a series for the square of sine, as given by the rule[52] *nihatya cāpavargeṇa* ...

$$\sin^2 x = x^2 - \frac{x^4}{\left(2^2 - \frac{2}{2}\right)} + \frac{x^6}{\left(2^2 - \frac{2}{2}\right)\left(3^2 - \frac{3}{2}\right)} - \ldots$$

Chapter VII of *Yuktibhāṣā* goes on to discuss different ways of deriving the *jīve-paraspara-nyāya*[53], which is followed by a detailed discussion of the cyclic quadrilateral. The chapter concludes with a derivation of the surface area and volume of a sphere.

Acknowledgement

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[51] *Yuktidīpikā* , cited earlier, p.118

[52] *Yuktidīpikā* , cited earlier, p.119.

[53] The relation between the sine and cosine of the sum or difference of two arcs with the sines and cosines of the arcs.

Ramanujan and Partial Fractions

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Abstract

The subject of partial fractions is usually confined to the standard calculus course and is viewed as a useful albeit mundane tool. This paper looks at partial fractions starting with Euler. We then consider some of the very surprising and appealing discoveries made by Ramanujan.

1 Introduction

The idea of partial fractions must surely have arisen after calculus and basic analysis had a solid footing. We can certainly find carefully prepared examples in Euler's Introduction in Analysis Infinitorum, Volume 1 [3, Chapter 2].

For example, [3, pp. 34–35] Euler proves that

$$\frac{1}{z^3(1-z)^2(1+z)} = \frac{1}{z^3}+\frac{1}{z^2}+\frac{2}{z}+\frac{1}{2(1-z)^2}+\frac{7}{4(1-z)}-\frac{1}{4(1+z)}, \quad (1.1)$$

or [3, pp. 31–32]

$$\frac{z^2}{(1-z)^3(1+z^2)} = -\frac{z-1}{4(1+z^2)}+\frac{1}{4(z-1)}-\frac{1}{2(z-1)^2}-\frac{1}{2(z-1)^3}. \quad (1.2)$$

Ramanujan was not the first to observe that there are appealing and sophisticated extensions of partial fractions. For example, there is the classical partial fraction expansion for the reciprocal of Jacobi's theta product [6, p.136]

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 - zq^n} = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-zq^{n-1})(1-q^n/z)} \tag{1.3}$$

$$= \frac{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}}{\sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n(n-1)}}, \tag{1.4}$$

where $|q| < 1$ and $z \neq q^{-N}$ for any integer N. Actually we find (1.3) in Ramanujan's Lost Notebook [5, p.1].

In the following pages, we shall examine some of Ramanujan's ideas. Our object will be to choose formulas that illustrate how one can proceed from first principles to obtain some quite surprising formulas.

2 A Partial Fraction

In the following, we shall use the standard notation [4, p.3, eq. (1.2.15) and p. 20, Ex. 1.2]. We shall always assume $|q| < 1$.

$$\begin{aligned}
(A)_n &= (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}), \\
(A)_\infty &= (A;q)_\infty = \prod_{n=0}^{\infty}(1-Aq^n), \\
\left[\begin{matrix} N \\ M \end{matrix}\right] &= \begin{cases} \frac{(q;q)_N}{(q;q)_M(q;q)_{N-M}}, & 0 \leqq M \leqq N \\ 0, & M < 0 \text{ or } M > N. \end{cases}
\end{aligned}$$

Our first result is the following:

Theorem 2.1.

$$\sum_{n=0}^{N} \left[\begin{matrix} N \\ n \end{matrix}\right] \frac{(q)_n q^{n^2}}{(z)_{n+1}(q/z)_n}$$

$$= \frac{1}{(1-z)(q)_N} + \sum_{n=1}^{N} \left[\begin{matrix} N \\ n \end{matrix}\right] \frac{(-1)^n q^{\frac{3}{2}n^2 + \frac{n}{2}}(q)_n}{(q)_{N+n}} \left(\frac{1}{1-zq^n} - \frac{1}{z-q^n}\right). \tag{2.1}$$

Proof. Both sides of (2.1) are rational functions of z with $2N+1$ simple poles at $z = q^{-N}, q^{-N+1}, \ldots, q^{-1}, 1, q, \ldots, q^N$. Let us calculate the residues of each side at these simple poles.

If $z = q^m$ for $1 \leqq m \leqq N$, then the residue on the right-hand side is

$$\begin{bmatrix} N \\ m \end{bmatrix} \frac{(-1)^{m-1} q^{\frac{3}{2}m^2 + \frac{m}{2}} (q)_m}{(q)_{N+m}}.$$

On the left-hand side

$$\begin{aligned}
& \lim_{z \to q^m} (z - q^m) \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_n q^{n^2}}{(z)_{n+1} (q/z)_n} \\
= & \sum_{n=m}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_n q^{n^2+m}}{(q^m; q)_{n+1} (q^{1-m}; q)_{m-1} (q)_{n-m}} \\
= & \sum_{n=m}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_n q^{n^2+m(m+1)/2} (-1)^{m-1}}{(q)_{n+m} (q)_{n-m}} \\
= & (q)_N \sum_{n=m}^{N} \frac{(-1)^{m-1} q^{n^2+m(m+1)/2}}{(q)_{n+m} (q)_{N-n} (q)_{n-m}} \\
= & (q)_N \sum_{n=0}^{N-m} \frac{(-1)^{m-1} q^{(n+m)^2+m(m+1)/2}}{(q)_{n+2m} (q)_{N-n-m} (q)_n} \\
= & \frac{(q)_N (-1)^{m-1}}{(q)_{2m} (q)_{N-m}} \sum_{n=0}^{N-m} \frac{(q^{-N+m})_n (-1)^n q^{n(n+1)/2+n(N+m)+m(3m+1)/2}}{(q^{2m+1})_n (q)_n} \\
= & \frac{(q)_N (-1)^{m-1} q^{m(3m+1)/2}}{(q)_{2m} (q)_{N-m}} \lim_{\tau \to 0} {}_2\phi_1 \left(\begin{matrix} q^{-N+m}, \tau^{-1}; q, \tau q^{N+m+1} \\ q^{2m+1} \end{matrix} \right) \\
& \qquad \text{(in the notation of [4, p.3, eq. (1.2.14)])} \\
= & \frac{(q)_N (-1)^{m-1} q^{m(3m+1)/2}}{(q)_{2m} (q)_{N-m} (q^{2m+1})_{N-m}} \\
& \text{(by the } q\text{-Chu-Vandermonde summation [4, p.11, eq. (1.5.2)])} \\
= & \begin{bmatrix} N \\ m \end{bmatrix} \frac{(-1)^{m-1} q^{m(3m+1)/2} (q)_m}{(q)_{N+m}}
\end{aligned}$$

Next we consider $z = q^{-m}$ for $0 \leqq m \leqq N$. In this case, the residue on the right-hand side is

$$\begin{bmatrix} N \\ m \end{bmatrix} \frac{(-1)^{m-1} q^{m(3m-1)/2} (q)_m}{(q)_{N+m}}.$$

On the left-hand side

$$
\begin{aligned}
&\lim_{z\to q^{-m}}(z-q^{-m})\sum_{n=0}^{N}\begin{bmatrix} N \\ n \end{bmatrix}\frac{(q)_n q^{n^2}}{(z)_{n+1}(q/z)_n}\\
= \; & -\sum_{n=m}^{N}\begin{bmatrix} N \\ n \end{bmatrix}\frac{(q)_n q^{n^2-m}}{(q^{-m})_m (q)_{n-m}(q^{m+1})_n}\\
= \; & \sum_{n=m}^{N}\begin{bmatrix} N \\ n \end{bmatrix}\frac{(q)_n q^{n^2+m(m-1)/2}(-1)^{m-1}}{(q)_m (q)_{n-m}(q^{m+1})_n}\\
= \; & (q)_N \sum_{n=m}^{N}\frac{(-1)^{m-1}q^{n^2+m(m-1)/2}}{(q)_{n+m}(q)_{N-n}(q)_{n-m}}\\
= \; & (q)_N \sum_{n=0}^{N-m}\frac{(-1)^{m-1}q^{(n+m)^2+m(m-1)/2}}{(q)_{n+2m}(q)_{N-n-m}(q)_n}\\
= \; & \frac{(q)_N(-1)^{m-1}q^{m(3m-1)/2}}{(q)_{2m}(q)_{N-m}}\sum_{n=0}^{N-m}\frac{(q^{-N+m})_n(-1)^n q^{n(n+1)/2+n(N+m)}}{(q^{2m+1})_n (q)_n}\\
= \; & \begin{bmatrix} N \\ m \end{bmatrix}\frac{(-1)^{m-1}q^{m(3m-1)/2}(q)_m}{(q)_{N+m}},
\end{aligned}
$$

as before.

Hence by the classical theorem for the representation of a proper rational function with simple poles by a partial fraction decomposition [2, pp.56–57], we have established Theorem 2.1. □

Note that the proof of Theorem 2.1 required nothing more sophisticated than the very elementary q-Chu-Vandermonde summation.

Theorem 2.1 has some very appealing corollaries. While it does not appear in Ramanujan's Lost Notebook, many instances of it do. It should be noted that this result was first explicitly stated by G.N. Watson [7, p.64].

Corollary 2.1.

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(z)_{n+1}(q/z)_n}=\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1-zq^n}. \tag{2.2}$$

Proof. Let $N \to \infty$ in Theorem 2.1. The quadratic exponent on q easily allows one to justify this limiting process, and a little algebra transforms the result into the above expression. □

From Corollary 2.1, G.N. Watson [7, p.64] and presumably Ramanujan deduced three important formulas for Ramanujan's third order mock-theta functions:

$$\begin{aligned} f(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, \\ \phi(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n} \\ \chi(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q)_n}{(-q^3;q^3)_n} \end{aligned}$$

Corollary 2.2. *[7, p. 64]*

$$f(q) = \frac{1}{(q)_\infty}\left(1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}\right), \tag{2.3}$$

$$\phi(q) = \frac{1}{(q)_\infty}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n q^{n(3n+1)/2}}{1+q^{2n}}\right), \tag{2.4}$$

$$\chi(q) = \frac{1}{(q)_\infty}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)q^{n(3n+1)/2}}{1-q^n+q^{2n}}\right) \tag{2.5}$$

Proof. To obtain (2.3) set $z = -1$ in (2.2) and simplify. To obtain (2.4) set $z = i$ in (2.2) and simplify. Finally to obtain (2.5) set $z = e^{\pi i/3}$. □

3 Another Partial Fraction

In this section we present a second result. This result is closely related to the one in the previous section as Watson has shown [7, p. 63–66] in that each is deducible from the same master formula. We treat this theorem separately because we want to deduce these results from very elementary identities.

Theorem 3.1.

$$\sum_{n=0}^{N}\begin{bmatrix} N \\ n \end{bmatrix}\frac{(q)_n q^{n^2+n}}{(zq^{\frac12})_{n+1}(q^{\frac12}/z)_{n+1}} = \sum_{n=0}^{N}\begin{bmatrix} N \\ n \end{bmatrix}\frac{(-1)^n q^{3n(n+1)/2}(q)_n}{(q)_{n+N+1}}\left(\frac{1}{1-zq^{n+\frac12}}+\frac{q^{n+\frac12}}{z-q^{n+\frac12}}\right) \tag{3.1}$$

Proof. We proceed exactly as we did with Theorem 2.1. Both sides of (3.1) are rational functions of z with $2N+2$ simple poles at $z = q^{-N-\frac12}, q^{-N+\frac12}, \dots, q^{-\frac12}, q^{\frac12}, \dots, q^{N-\frac12}, q^{N+\frac12}$. We now calculate the residues of each side at these simple poles. Furthermore it is easy to check that each side is symmetric in z and $\frac1z$; hence we need only check the residues at the positive power of q.

If $z = q^{m+\frac12}$ for $0 \leqq m \leqq N$, then the residue on the right-hand side is clearly

$$\begin{bmatrix} N \\ m \end{bmatrix}\frac{(-1)^m q^{3m(m+1)/2+m+\frac12}(q)_m}{(q)_{m+N+1}}.$$

On the left-hand side

$$\begin{aligned}
&\lim_{z\to q^{m+\frac12}}(z-q^{m+\frac12})\sum_{n=0}^{N}\begin{bmatrix} N \\ n \end{bmatrix}\frac{(q)_n q^{n^2+n}}{(zq^{\frac12})_{n+1}(q^{\frac12}/z)_{n+1}}\\
=\ &\sum_{n=m}^{N}\begin{bmatrix} N \\ n \end{bmatrix}\frac{(q)_n q^{n^2+n+m+\frac12}}{(q^{m+1})_{n+1}(q^{-m})_m(q)_{n-m}}\\
=\ &(-1)^m q^{m(m+1)/2+m+\frac12}(q)_N\sum_{n=m}^{N}\frac{q^{n^2+n}}{(q)_{n+m+1}(q)_{N-n}(q)_{n-m}}\\
=\ &(-1)^m q^{m(m+1)/2+m+\frac12}(q)_N\sum_{n=0}^{N-m}\frac{q^{(n+m)^2+n+m}}{(q)_{n+2m+1}(q)_{N-n-m}(q)_n}\\
=\ &\frac{(-1)^m q^{m(m+1)/2+m+\frac12+m^2+m}(q)_N}{(q)_{2m+1}(q)_{N-m}}\\
&\times\sum_{n=0}^{N-m}\frac{(q^{-N+m})_n(-1)^n q^{n(n+1)/2+n(N+m+1)}}{(q)_n(q^{2m+2})_n}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^m q^{\frac{3}{2}m^2+\frac{5m}{2}+\frac{1}{2}}(q)_N}{(q)_{2m+1}(q)_{N-m}} \lim_{\tau\to 0} {}_2\phi_1\left(\begin{matrix} q^{-N+m}, \frac{q}{\tau}; q, \tau q^{N+m+1} \\ (q)^{2m+2} \end{matrix}\right) \\
&= \frac{(-1)^m q^{\frac{3}{2}m^2+\frac{5m}{2}+\frac{1}{2}}(q)_N}{(q)_{2m+1}(q)_{N-m}(q^{2m+2})_{N-m}} \\
&\qquad \text{(by the } q\text{-Chu-Vandermonde summation [4, p. 11, eq. (1.5.2)])} \\
&= \left[\begin{matrix} N \\ m \end{matrix}\right] \frac{(-1)^m q^{3m(m+1)/2+m+\frac{1}{2}}(q)_m}{(q)_{m+N+1}}
\end{aligned}
$$

As with Theorem 2.1, our theorem follows from the standard partial representation of a proper rational function with simple poles [2, pp. 56–57]. ☐

Corollary 3.1. *[7, p.66]*

$$
\sum_{n=0} \frac{q^{n^2+n}}{(zq^{\frac{1}{2}})_{n+1}(q^{\frac{1}{2}}/z)_{n+1}} = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1 - zq^{n+\frac{1}{2}}}. \tag{3.2}
$$

Proof. Let $N \to \infty$ in Theorem 4.1. As in the proof of Corollary 2.2, the quadratic exponents on q easily allow one to justify this limit. Algebraic simplification then yields the assertion in Corollary 3.1. ☐

From Corollary 3.1, we can easily deduce (as did Watson [7, p. 66] and probably Ramanujan) three more important formulas for three more of the third order mock-theta functions:

$$
\begin{aligned}
\omega(q) &= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}, \\
\upsilon(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}, \\
\rho(q) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n(n+1)}}{(q^3;q^6)_{n+1}}.
\end{aligned}
$$

Corollary 3.2. *[7, p.66]*

$$\omega(q) = \frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{3n(n+1)}\frac{1+q^{2n+1}}{1-q^{2n+1}}, \tag{3.3}$$

$$\upsilon(q) = \frac{1}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{3n(n+1)/2}\frac{(1-q^{2n+1})}{(1+q^{2n+1})}, \tag{3.4}$$

$$\rho(q) = \frac{1}{q^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{3n(n+1)}\frac{(1-q^{4n+2})}{1+q^{2n+1}+q^{4n+2}}. \tag{3.5}$$

Proof. To obtain (3.3) replace q by q^2 then set $z = 1$ in (3.2), and simplify. To obtain (3.4), set $z = i$ in (3.2) and simplify. To obtain (3.5) replace q by q^2, then set $z = e^{2\pi i/3}$ in (3.2) and simplify. □

4 The Simplest Partial Fraction

In the last two sections. we proved two partial fraction decompositions relying only on the q-Chu-Vandermonde summation. In this section we provide a proof of (1.3) that does not even require that much.

Theorem 4.1.

$$\frac{(q)_N}{(z)_{N+1}(q/z)_N} = \frac{1}{(1-z)(q)_N} + \sum_{n=1}^{N}\begin{bmatrix} N \\ n \end{bmatrix}\frac{(q)_n(-1)^n q^{n(n+1)/2}}{(q)_{n+N}}\left(\frac{1}{1-zq^n} - \frac{1}{z-q^n}\right) \tag{4.1}$$

Proof. Both sides of (4.1) are rational functions of z with $2N+1$ simple poles at $z = q^{-N}, q^{-N+1}, \ldots, q^{-1}, 1, q, \ldots, q^N$.

First we calculate the residue at $z = q^m, 1 \leqq m \leqq N$. The residue on the right-hand side is clearly

$$(-1)^{m-1}\begin{bmatrix} N \\ m \end{bmatrix}\frac{(q)_m q^{m(m+1)/2}}{(q)_{m+N}}.$$

On the left-hand side, the residue is

$$\begin{aligned} \lim_{z \to q^m} \frac{(z-q^m)(q)_N}{(z)_{N+1}(q/z)_N} &= \frac{q^m (q)_N}{(q^m)_{N+1}(q^{1-m})_{m-1}(q)_{N-m}} \\ = \frac{(-1)^{m-1} q^{m(m+1)/2}(q)_N}{(q)_{N+m}(q)_{N-m}} &= (-1)^{m-1} \begin{bmatrix} N \\ m \end{bmatrix} \frac{(q)_m q^{m(m+1)/2}}{(q)_{m+N}}. \end{aligned}$$

Next we consider $z = q^{-m}$, $0 \leqq m \leqq N$. The residue on the right-hand side in this instance is

$$-q^{-m} \begin{bmatrix} N \\ m \end{bmatrix} \frac{(q)_m (-1)^m q^{m(m+1)/2}}{(q)_{m+n}} = (-1)^{m-1} \begin{bmatrix} N \\ m \end{bmatrix} \frac{(q)_m q^{m(m-1)/2}}{(q)_{m+N}},$$

and on the left-hand side, the residue is

$$\begin{aligned} \lim_{z \to q^{-m}} \frac{(z-q^{-m})(q)_N}{(z)_{N+1}(q/z)_N} &= \frac{-q^{-m}(q)_N}{(q^{-m})_m (q)_{N-m}(q^{m+1})_N} \\ = \frac{(-1)^{m-1} q^{m(m-1)/2}(q)_N}{(q)_{N+m}(q)_{N-m}} &= (-1)^{m-1} \begin{bmatrix} N \\ m \end{bmatrix} \frac{(q)_m q^{m(m-1)/2}}{(q)_{N+m}}. \end{aligned}$$

As before, our result now follows from the standard theorem on partial fraction decompositions [2, pp.56–57] □

Corollary 4.1. *Identity (1.3) is valid.*

Proof. The same argument used for Corollaries 2.1 and 3.1 holds here as well. □

5 Conclusion

There are many more results beyond those considered in this paper that were included in Ramanujan's Lost Notebook. Bruce Berndt and I [1] are publishing a full treatment of the results from the Lost Notebook, and we present in Chapter 12 of that work a full account of Ramanujan's partial fraction theorems. The proofs there are more succinct, but rely on more sophisticated background. The object in this paper was to illustrate the fact that one can deduce some of Ramanujan's most surprising partial fraction formulas using nothing deeper than the q-analog of the Chu-Vandermonde summation.

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Contributions of Indian Mathematicians to Quantum Statistics

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Abstract

While modern Algebra and Number Theory have well documented and established roots deep into India's ancient scholarly history, the understanding of the springing up of statistics, specifically quantum statistics, demands a closer inquiry. My project is two-fold. Firstly, I explore and delineate the cultural and educational circumstances that presided over the inception of the very concept that quantum theory required its own dedicated statistical analysis. My quest therefore is anchored in a brief review of the pioneering contributions of personalities as diverse as those of Bose and Chandrasekhar, or Raman and Krishnan, and Mahalanobis. Secondly, I examine how the intellectual climate and some of the local mathematical traditions have fostered the ongoing development of quantum probability and stochastic processes theories in India.

> *Science is not an impersonal stream of discoveries. It is created by human beings and its advances are very much products of highly personal actions and reactions of some gifted individuals.* [70, p.31]
> *... science is a collective endeavour ... any single life is but a fragment in a larger fabric.* [71, p.55]

1 Preliminary considerations

The main questions I want to explore in this essay are *How* and *Why* an interest in quantum statistics developed among Indian mathematicians. Indeed, the field of quantum statistics might appear at first sight to be a singularity against the background of profound traditions, the deep roots of which are illustrated in other communications reported in these proceedings on the History of Indian Mathematics. Upon listening to these enlightening contributions, we could not but be vividly reminded of the continuity between the ancient schools and modern disciplines such as algebra, geometry, combinatorics and number theory, in which the accomplishments of Indian mathematicians are recognized the World over as centrally influential.

In contrast, quantum probability, and in particular the theory of quantum stochastic processes, is a recent field of enquiry in which considerable pioneering work was achieved in twentieth–century India. It would be futile to attempt and give here a detailed survey that would do justice to all facets of this effort. Rather, I will focus on those which relate most to areas of mathematical physics with which I am familiar, and yet bring out the particular flavours of the Indian contributions and of the intellectual climate in which they originated.

Among foreign physicists and mathematicians, the very words of "quantum" and "statistics" might evoke developments in Indian science of cultures imported here from the outside in the early twentieth–century. Such a naive cosmopolitan account would however miss several Indian idiosyncrasies, two extremes among which are: a form of governmental establishment that encourages the influence of strong personalities, and yet a demographic diversity that allows the independent pursuits of individual thinkers. To illustrate this dualism, suffice it to mention here only one of its many aspects: the flowering in the nineteenth century of the "Bengali Renaissance" right in Calcutta, while the port-city was the traditional seat of British presence.

I will thus sketch in Section 2 how strong personalities developed into pioneers who paved the way towards the modern form of quantum statistics: S.N. Bose, C.V. Raman and K.S. Krishnan, S. Chandrasekhar, and P.C. Mahalanobis. In section 3, I turn to contemporary Indian figures who have contributed to the developments of various aspects of the theory of quantum probability and of quantum stochastic processes: V.S. Varadarajan, K.R. Parthasarathy, K.B. Sinha, and M.D. Srinivas. In the interest of conciseness, the bibliography at the end of this essay is intended to be an integral part of my account, as the titles of the papers I included provide in a nutshell the most essential information their authors surely wanted to convey. I should also mention that in the course of my essay I respect the contemporary usages when speaking of cities like Calcutta [Kolkata] or Madras [Chennai].

2 The pioneers

2.1 Satyendra Nath Bose, 1894–1974

Wearing an illustrious Bengali patronym, Bose was born and died in Calcutta, studied there, receiving his M.Sc. from Presidency College in 1915 and continuing his studies in mathematics and physics at the University College of the University of Calcutta, taking his Ph.D. in 1917; he first taught there, until he went to Dacca in 1921, from which he came back to Calcutta in 1945 to teach again there until 1956.

After he had translated and published a collection of Einstein papers on relativity, he spent time rethinking ("in my own way" as he declared) the published lectures of Boltzmann and of Planck on classical statistical mechanics, as well as the the book of Gibbs, and several of the original papers of Einstein, Bohr and Sommerfeld, i.e. the foundations of a well–informed study of the current physical literature, namely what we call today the "old" quantum mechanics.

Bose's own contributions to quantum statistics [10, 11] occurred during a burst of creativity that lasted just a few years. This happened despite the well–documented and enthusiastic support of Einstein who was instrumental in the granting to Bose of a generous two–year travel fellowship that brought him into contact with Langevin and Marie Curie in Paris, and with Einstein and the Berlin establishment. After this, Bose returned to teaching and some research, first in Dacca and then again in Calcutta.

The reason for Einstein's enthusiasm must have been the authentic simplicity of Bose's derivation of the Planck black–body radiation law which is best understood when written in the form:

$$\rho(\nu,T) = n(\nu)\cdot u(\nu,T)$$

with

$$n(\nu) = \frac{8\pi\nu^2}{c^3} \quad \text{and} \quad u(\nu,T) = \frac{h\nu}{\exp(h\nu/kT)-1} \tag{2.1}$$

where ρ is the energy density distribution of the radiation, as a function of the frequency ν and the temperature T at which the radiation is in thermal equilibrium. Earlier, the first factor, $n(\nu)$, had been derived from a classical notion, the number $N(\nu) = V n(\nu)$ of electromagnetic modes of frequency ν; V is the volume of the cavity in which the radiation is trapped. This number obtains as a straightforward adaptation of an argument by Rayleigh (1877–78) on standing sound waves, the only two changes required here being to substitute the speed of light c for the speed of sound; and to take into account that electromagnetic waves are transversal waves, thus with two polarizations at the same frequency, while sound waves in air are compression waves, thus with only one mode for each frequency. The second term, $u(\nu,T)$, had appeared as a "lucky" manifestation of the genius of Planck who, in 1900, interpolated between previously known formulas valid separately in low and high frequency regimes. In the intervening years, Einstein had proposed that the radiation consists of *indivisible* light quanta of energy $h\nu$, an interpretation he supported by his concise and consistent explanation of all the known features of the photo-electric effect, features which the Maxwell theory of electromagnetism had thoroughly failed to account for. The photon hypothesis enabled Debye and Einstein to account for $u(\nu,T)$ and Bose was willing to accept their quantum counting argument. It was on the first term that he concentrated his critique; we read from his famous June 4, 1924 letter to Einstein, that "I have tried to deduce the coefficient $8\pi\nu^2/c^3$ in Planck's law independent[ly] of the classical electrodynamics, only assuming that the ultimate elementary region in the phase space has the content h^3." In postulating the irreducibility of any such partition, Bose was striving to restore the consistency of the premises; both terms in the Planck radiation formula were to have a purely quantum origin: now, the *Bose–Einstein statistics* had found a proper logical frame.

Einstein was sufficiently impressed by Bose's manuscripts to translate them into German and arrange for their publication. Moreover, he assimilated Bose's two papers thoroughly enough to realize that the new statistics, if extended from radiation to material particles would imply that a *quantum* ideal gas [23] would condensate in a superfluid phase if the temperature were lowered deeply enough: a phenomena that has no classical analogue.

Although Einstein was not fully satisfied, Bose was not able to respond to Einstein's objections, even after meeting personally with him in Berlin. How Dirac was able to perform the necessary synthesis through his symmetrization prescription has been described in [66] and [41, pp.424–425]; incidentally, the word "Boson" to refer to particles that obey the Bose–Einstein statistics was coined by Dirac.

As long as liquid Helium remained the only available laboratory manifestation of the Bose–Einstein condensation, a hiatus remained between theory and experiment, since the intermolecular interactions surely present in superfluid Helium were not accounted for in the original model of an *ideal* quantum gas. The situation changed drastically in the 1980s and 1990s with the observation of Bose–Einstein condensation in dilute atomic gases at temperatures in the milli– and then micro–Kelvin ranges, thus projecting again Bose's name to the frontier of contemporary physics.

2.2 Chandrasekhara Venkata Raman, 1888–1970 and Kariamanikkam Srinivas Krishnan, 1898–1961

Raman's 1930 Nobel Prize serves here as a historical marker to witness the development of modern physics in India at the time when the new quantum statistics was emerging in the midst of significant developments in the classical theory of stochastic processes. The Nobel citation reads *for his work on the scattering of light and for the discovery of the effect named after him.* It was only in February 1928 [52] that the discovery was made by Raman's research team at the Indian Association for the Cultivation of Science; he was Professor of Physics at the University of Calcutta since 1917 until 1933 when he joined the Indian Institute of Science in Bangalore; there, in 1948, he became director of the Research Institute that bears his name. His formal education had ended in 1907 with a M.A. from the University of Madras; he had obtained a B.A. from Madras Presidency College in 1904, at the age of 16.

The Raman effect belongs to molecular spectroscopy, and it helps determine the structure of molecules. Specifically, when a spectroscope analyses the light emitted from a transparent medium on which a monochromatic light beam of frequency ν_o is directed, one observes symmetrically on each side of the original spectral line a relatively faint secondary spectrum

$$\{\nu = \nu_o \pm \nu_m\} \ , \tag{2.2}$$

now called the *Raman spectrum.* The observed frequency shifts ν_m and the relative intensities of the secondary lines – both of which are characteristic of the material target rather than of the incident light beam – allow us to interpret this spectrum as being due to the inelastic collisions between the incident light quanta and the molecules, involving the discrete energy levels $h\nu_m$ associated by quantum theory to the vibrational and/or rotational degrees of freedom of the molecules of the medium – primarily liquids, but also solids and even gases. The fact that the so-called "Stokes" lines (those on the low-frequency side of the exciting line) are more intense than the corresponding "anti–Stokes" lines (on the high-frequency side) in the ratio

$$I_a/I_s = \exp(-h\nu_m/kT) \tag{2.3}$$

is taken as a further confirmation of the quantum nature of the Raman effect, so much so that to the physicist R.W. Wood is attributed the opinion that *"Raman's long and patient study of the phenomenon of light scattering is one of the most convincing proofs of the quantum theory"* [74, p.208], thus suggesting it was on par with Einstein's explanation of the photo-electric effect. As R.W. Wood may not be so well known in the mathematical community, let it be noted that Niels Bohr, who had recommended Raman for the 1930 Nobel Prize for Physics, not only had also included Wood's name as a co-nominee, but had done so even earlier for the 1929 Prize [54].

While the effect had been predicted before by Smekal (1923), its actual discovery demonstrated a degree of sophistication, both theoretical and experimental, that was immediately recognized – in particular by the Nobel committee – as a crowning achievement of what we call now the "old" quantum physics epitomized by Bohr. All along, Raman continued to be very much involved in more general, albeit classical, physical aspects of oscillatory phenomena. Indeed two examples ought to be mentioned here; the first is that along with his better known activities in optics, Raman maintained a creative interest in acoustics; see for instance [51] where Raman describes the physics of musical instruments ... including some Indian instruments such as the *sitar* and the *tabla.* The second example of Raman's activity is closer to the subject of this essay: after having been struck by the deep blueness of the Mediterranean sea, he questioned Rayleigh's theory on the blueness of the sky, and he applied to the diffraction of light the Einstein–Smolukowski stochastic theory of fluctuations and the attendant correlations [50], which was extended by his collaborators; amongst them was K.S. Krishnan, who belongs to this essay on several accounts. He performed some of the crucial experiments leading to the unambiguous detection and thus the very discovery of the Raman effect; and later, in the early 1940s, when as professor of physics at Allahabad, Krishnan kindled Harish–Chandra's first scientific direction of research – theoretical physics – which Harish–Chandra pursued in Bangalore with Bhabha [5], and then in Cambridge under Dirac [71, p.56].

Krishnan's talents as an experimental physicist first came to light in Raman's IACS laboratory; recall [52] and see [54]. The cultivation of these exceptionally acute abilities later made him famous as a pioneer in the investigation of quite an array of diverse phenomena in condensed matter physics – cf. e.g. [4] – so much so that the present address of the National Physics Laboratory, of which he was the first director, is on K.S. Krishnan Road in Delhi. His early association with Raman's research programme on "light scattered in a homogeneous medium" is explicitly acknowledged in the opening paragraphs of both [35, 36], his only overt contribution in and to mathematics. This belongs here for three reasons: first, it stems from an analysis of a physically important stochastic process; second, it gives a mathematically elegant proof for a special version of a theorem now known to all probability theorists; and third, it provides my story with a link to Ramanujan, indispensable to any paper in the history of modern Indian mathematics.

First, the theorem. Krishnan begins by citing [22] – in which Einstein pursues the theory of critical opalescence proposed by Smolukowski [58] – for its use of an approximation amounting to the formula:

$$\lim_{\alpha\to 0}\ \alpha \sum_{n=-\infty}^{+\infty} \frac{\sin^2(n\alpha+\theta)}{(n\alpha+\theta)^2} = \int_{-\infty}^{\infty} dx \frac{\sin^2 x}{x^2}\ . \qquad (2.4)$$

What Krishnan realizes is that this formula hides a deeper mathematical truth, namely the following general result which he proceeds to prove. Let f be an even real function of a real variable x such that its Fourier transform

$$\tilde{f}(k) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x) \exp(-ikx)$$

exists and vanishes for $|k| \geq k_o$. *Then*

$$\forall\, \alpha \in (0, 2\pi/k_o] \quad : \quad \alpha \sum_{n=-\infty}^{+\infty} f(n\alpha + \theta) = \sqrt{2\pi}\,\tilde{f}(0) \quad . \qquad (2.5)$$

Note that: (i) $\sin^2 x/x^2$ satisfies the assumptions, with $k_o = \sqrt{2\pi}$; (ii) the validity of formula (2.5) extends over a range of values of α rather than as a limit, as (2.4) does; (iii) the assumptions are very permissive; and (iv) (2.5) itself is a special case of the *sampling theorem*:

$$\forall\, \alpha \in (0, 2\pi/(k_o + k)] \quad : \quad \alpha \sum_{n=-\infty}^{+\infty} f(n\alpha) \exp(-in\alpha k) = \sqrt{2\pi}\,\tilde{f}(k) \qquad (2.6)$$

variously ascribed to Nyquist, Shannon or Whittaker; probabilists learn this from [25, Thm.XIX.5.2]; and it is also recognized to be of cardinal relevance in communication theory.

The second aspect of (2.5), which Krishnan recognizes already in [35], is that the functions f for which the formula is valid cover a wide range; from a wealth of examples, he is manifestly gratified in [36] to choose and present, in the light of his own result (2.5), several of Ramanujan's identities.

This spontaneous demonstration of such an interdisciplinary purview appears to have been a major contributing factor in the systematic development of the modern scientific establishment in India. Indeed several other scientists of the time shared this broad vision. Consider thus briefly the case of H.J. Bhabha (1909–1966), whom I mentioned already in connection with Harish-Chandra [5]. Along with his long-lasting interest in relativistic quantum theory of elementary particles [6], he belongs to the present story also for his classical stochastic analysis of electron cascades [7, 8], a theoretical aspect of cosmic rays physics; furthermore – as with Krishnan – experimental physicists know him perhaps even better for his persistent and successful observational pursuits. Bhabha's multidimensional drives are further evidenced by his being the founder in 1945, and first Director, of the Tata Institute for Fundamental Research in Bombay; and by the eminent role he played on the international political scene as president of the IAEC for the peaceful use of nuclear power. Beyond the confines of this essay, I discern in [28] several elements that would help enlighten a discussion of the considerations (or theses) I proposed in section 1 above; see also subsection 2.4 below.

2.3 Subrahmanyan Chandrasekhar, 1910–1995

Chandrasekhar is also an alumnus of Presidency College, Madras (B.Sc. with honours in Physics, 1930), and then Trinity College, Cambridge where he studied under R.H. Fowler, met Dirac, and was awarded the Ph.D. in 1933. After Cambridge, he was associated with the University of Chicago from 1937 to the end of his life, thus overlapping there with the Marshall Stone era in mathematics. Although many of us know Chandrasekhar as a most agile classical applied mathematician, he is known primarily as an astronomer and was Editor of the Astrophysical Journal from 1952 to 1971. He received the Nobel Prize in Physics in 1983, which brings us straight into his early involvement with quantum statistics.

Chandrasekhar was prompted in this direction by no less a master than Sommerfeld during an apparently dramatic private interview that took place in the fall of 1928 in Madras, in the course of which the "Professor of Professors" assured the eighteen year old student – so proud of having studied on his own Maxwell–Boltzmann statistics and an early edition of Sommerfeld's book *Atomic Structure and Spectral Lines* – that everything he knew had been superseded by the new quantum mechanics of Schroedinger, Heisenberg, and Dirac. To drive home his point, Sommerfeld mercifully passed to Chandrasekhar the galley–proofs of a paper where he proposed to apply the Fermi–Dirac statistics to the theory of electrons in metals. Chandrasekhar had thus inadvertently landed on the hinge between the "old" quantum mechanics and the "new", with a clear admonition that the passage from the former to the latter was ineluctable. "Chandra would later characterize this encounter as the 'single most important event' in his scientific career." [75, p.1]

Chandrasekhar indeed understood the lesson: he soon wrote a paper of his own [15]; and he read a 1926 paper by R.H. Fowler, which he describes later as "the fundamental discovery that the electron assembly in the white dwarfs must be degenerate in the sense of the Fermi–Dirac statistics" [17, *fn.*1,p.451]: indeed Fowler had identified the electron degeneracy pressure as the pressure that keeps the stars from gravitational collapse. Following up on this reading, Chandrasekhar published a string of papers on *relativistic* quantum degenerate stellar configurations and the theory of white dwarfs [16]; these early papers are part of his Nobel prize citation.

It is in the course of this analysis that Chandrasekhar made the discovery that white dwarfs have a maximum mass $M_{max} \simeq 1.4\, M_{\odot}$ where $M_{\odot}$ is the mass of the sun:

$$M_{max} = (\frac{hc}{G})^{3/2} {m_B}^{-2} \qquad (2.7)$$

where m_B refers here to the mass of electrons (or, see below, neutrons). Note that the coefficients in front of ${m_B}^{-2}$ depend only on universal physical quantities: the Planck constant h, the velocity of light c, and the gravitational constant G. The maximum mass M_{max} is called the *Chandrasekhar limit.*

Although Chandrasekhar's derivation is mathematically correct, it seemed somewhat cumbersome. Already by the end of 1932, L.D. Landau not only presented a more elementary explanation of the Chandrasekhar limit but, upon hearing of the discovery of the neutron, he applied the above formula to then putative neutron stars. Chandrasekhar's analysis implies that stars of mass larger than the critical mass M_{max} could not be self-sustained, but would collapse under their own gravity. These theoretical predictions had to wait till the 1960s for observational confirmation; in the meantime, these consequences of the Fermi–Dirac statistics first caused quite a surf, at the forefront of which stood Eddington, an astronomer commanding an imposing authority, who is reported to have spoken of a *reductio ad absurdum* calling for the interposition of an as yet unknown fundamental theory. For the unfolding of the resolution of the conflict, cf. e.g. [53], the title of which already indicates the complete extent to which Chandrasekhar was ultimately vindicated. Chandrasekhar however did not pursue this line of research past the synthesis he presented in [17]) (see in particular Chap. XI). Nevertheless, he revisited in the mid–1960s the question of the general–relativistic stability of stars.

The early period of Chandrasekhar's diverse activities – the most pertinent to my purpose here – calls for three remarks relative to the emergence of quantum statistics in India. The first remark is that while [15] – accepted upon Fowler's recommendation – is widely acknowledged as Chandrasekhar's 'first' paper on the subject, it was preceded by an earlier paper, really his first publication, which had appeared in volume 3 (1928) of the *Indian Journal of Physics;* its very title indicates already the astronomical motivations that were to move Chandrasekhar along much of his career: it announces unambiguously his interest in the "Thermodynamics of the Compton Effect" (compare with [15]) "with Reference to the Interior of the Stars."

The second remark is to draw attention to the mention, in the title of [15] of the 'New Statistics': in the matter of a few months since his encounter with Sommerfeld recalled above, Chandrasekhar had caught the gist of the Fermi–Dirac statistics. The Fowler–Chandrasekhar papers truly rank among the very first and perhaps the most spectacular applications of the Fermi–Dirac statistics, right with the semi–classical theory of multi–electron atoms proposed independently in 1927-1928 by Thomas and by Fermi (who had proposed in 1926 the statistics that bears his name); and the 1928 Sommerfeld theory of electrons in metals. The impact of the discovery of the Chandrasekhar limit may be compared with the prediction of the superfluid condensate in perfect quantum gases obeying the Bose–Einstein statistics (see Subsection 2.1. above).

My third remark concerns the way Chandrasekhar later considered quantum statistics: "It is somewhat misleading to use the word 'statistics' in 'Einstein–Bose statistics' and 'Fermi–Dirac statistics.' There is only one statistics, namely the Gibbs statistics ... the symmetrical and antisymmetrical cases simply correspond to two different assumptions ... but nevertheless, we have the same statistical theory (Gibbs) underlying both cases. It would be more logical to refer to 'Einstein–Bose formulae' and 'Fermi–Dirac formulae'." [17, p.384, fn.5]. The reason for this footnote seems to have lost some of its bite today, perhaps partly in view of the spin–statistics theorem of quantum field theory; another reason might be more idiosyncratic: *classical* mechanics and *classical* statistics were to take an increasingly more important place in Chandrasekhar's career; cf. for instance the masterful paper [18] that he reportedly had first prepared for his own use and published only at the insistence of von Neumann. I find it interesting to note here, from [65, p.i], that during the summer of 1928 Chandrasekhar had visited his uncle C.V. Raman in Calcutta, who was active on both fronts of physics, quantum and classical: the Raman effect had just been discovered, *and* the Einstein–Smolukowski theory of fluctuations had recently been called upon – and refined – to account for light scattering in fluids; see Subsection 2.2 above. Perhaps in a belated reflection of this early encounter, Chandrasekhar selected in [19] " ... a series of papers which Smolukowski wrote during the last five years of his life, papers in which the foundations of the modern theory of stochastic processes were laid ..."

2.4 Prasanta Chandra Mahalanobis, 1893–1972

As the eminence of statistical studies in India is inseparable from Mahalanobis' name, it may be worth mentioning, in an essay on the contributions of Indian mathematicians to quantum statistics, that Mahalanobis himself graduated in 1912 with honours in *physics* from Presidency College in Calcutta, and that he was the only candidate to achieve First Class in Physics when he took the tripos in Cambridge. Furthermore, it was from within the Physics department of Presidency College in Calcutta that, in the early 1920s, Mahalanobis planted a seed, the Statistical Laboratory, that under his tutelage became in 1931 the Indian Statistical Institute, which in 1959 was officially declared an "Institution of National Importance" by an act of Parliament. Indeed, to ensure that a flow of new ideas and problems would stimulate its team of statisticians, the Institute included, from its beginning, practitioners of other disciplines independently organized in a wide variety of departments; for the immediate purpose of this essay, suffice it to recall the strong tradition maintained in their own fields by the ISI mathematicians and physicists who personify the motto of the Institute "Unity in Diversity".

In the public-relations material distributed by ISI, one reads that Mahalanobis was a science organizer who disliked all forms of bureaucracy in science. Be that as it may, the inheritance of this ideal may begin to explain how our Indian colleagues manage to maintain their personal scholarly production in difficult interdisciplinary fields, amidst often demanding circumstances and extensive administrative responsibilities. In this sense also, Mahalanobis ought therefore to be mentioned here as one of the pioneers who created a home for the educational and research initiatives discussed in the next section. As unmistakable signs that Mahalanobis' successors have successfully pursued his lofty ideals I ought to mention two of them who are directly related to my story: C.R. Rao, who lists among his more than 50 Ph.D. students both Varadarajan (1960) and Parthasarathy (1962), the contributions of whom I delineate in subsections 3.1 and 3.2 below; and the current ISI Director, Kalyan B. Sinha, cf. subsection 3.3.

3 Contemporary Figures

3.1 Veeravalli S. Varadarajan

In the 1950s and 1960s the frustrating inconsistencies of quantum field theory prompted fundamental re-examinations of the mathematical foundations of quantum mechanics laid down in 1932 by von Neumann. The henceforth universal Fock space framework – that imprudently extended to infinitely many degrees of freedom von Neumann's uniqueness theorem on the representation of the canonical commutation relations for finitely many degrees of freedom – was the first casualty of this onslaught, as the works by Friedrichs, and by Garding and Wightman, unearthed the existence of *uncountably* many inequivalent irreducible representations of the CCR for infinitely many degrees of freedom.

This discovery was part of a renewed interest in earlier proposals to revise the basic axioms of quantum mechanics, namely those by Jordan, von Neumann and Wigner (1934) and by Birkhoff and von Neumann (1936).

JvNW's "On an Algebraic Generalization of the Quantum Mechanical Formalism" is an attempt to replace the algebra of all bounded operators on a Hilbert space, which ends up with a complete classification of all finite–dimensional Jordan algebras; this approach matured in the hands of Segal (1947), and of Haag and Kastler (1964) into the C^*-algebraic approach to quantum field theory and statistical mechanics; the conceptual problem of characterizing the Jordan algebras, the elements of which are the self-adjoint elements of a C^*-algebra, was solved in the 1980s by Alfsen and his collaborators.

BvN's "The Logic of Quantum Theory" focuses on the structure of the projectors on a Hilbert space, and is perhaps better characterized as the germ of a non-Boolean proposition calculus or, even better, of an infinite–dimensional projective geometry.

This is where Varadarajan enters the story. A mathematician by training – M.A. in mathematics and statistics, University of Madras, 1957; Ph.D. in mathematics, ISI–Calcutta, 1960 – he made a point "to mention that it was through the lectures of Prof. G.W. Mackey at Seattle in 1961 on the mathematical foundations of quantum mechanics that I first began to appreciate the beauty and depth of the subject" [69]. As a complement to this tribute to Mackey's influence on the coming generation of mathematicians and mathematical physicists throughout the world, let me mention also that at the very same time Varadarajan was imbibing the physical thinking underlying Mackey's project, some of us were learning its mathematical underpinnings – especially systems of imprimitivity and induced representations (on which Varadarajan was already an expert) – starting from Mackey's mimeographic Chicago lecture notes [38] that were circulating at CERN in Geneva and elsewhere in Europe.

Moving forward in time, one discerns that the overarching theme of Varadarajan's many-faceted research – much of which conducted at UCLA – came to be the mutually stimulating roles geometry, and in particular symmetry, plays in mathematics and in physics.

In mathematics, his efforts were directed first to measure-theoretical aspects of, and convergence problems in classical probability, a field in which he wrote an early burst of some fifteen papers in a span of five years; his research turned then to the representation theory of Lie groups and algebras, its first published result being [47]. He later established himself as one of the closest followers, the best interpreter and heir of Harish–Chandra who, after a presentation by Varadarajan, indeed told him that "he was very pleased ... and felt like a composer meeting a conductor." [70, p.32]. Varadarajan's mathematical purview extends as well to the theory of fiber bundles and connections, cf. e.g. [73] in which the physics reader is treated to a grand tour from Weyl and

Dirac, through Bohm-Aharanov, to Yang-Mills and Atiyah and Bott, not to speak of Singer; cf. also [72].

Varadarajan's interest in the foundations of quantum mechanics is already evidenced in [67] and continues to this day [3, 12]. With regard to quantum statistics proper – the focus of this essay – the projective geometry approach Varadarajan chooses in [68] allows him to offer what I consider to be the best presentation of a central element in the interpretation of von Neumann density matrices as quantum states, namely Gleason's theorem [68, Thm.4.23] and its extension to probability measures on the lattice of projectors of a general von Neumann algebra [68, pp.146-147]; the latter thus establish contact, at the representation level, with the other branch of quantum axiomatics, the C^*-algebraic approach (see above).

3.2 Kalyanapuram Rangachari Parthasarathy

Shortly after Varadarajan (see subsection 3.1 above), Parthasarathy obtained his Ph.D. from ISI–Calcutta in 1962, after having taken the BA from the University of Madras in 1956. They actually wrote together [47]. With certainly different emphases, they both had acquired a broad background in functional analysis allowing them to master measure theory and integration, Banach spaces and involutive algebras, and representations of Lie groups and algebras. Pertaining more immediately to Parthasarathy's training in probability, I wish to note his initial stay abroad, at the Steklov Mathematical Institute in Moscow (1962-1963).

Early confluences, bearing Parthasarathy own stamp, obtained in his classic/classical [42], and his sharp but perhaps somewhat less widely known [43]. The collection [9] illustrates the diverse interests Parthasarathy developed during his career.

Several of these strands are later braided in his magnum opus [44], the official birth certificate of the theory of *quantum stochastic processes;* for a presentation offering additional perspectives, see also [45].

Three questions must be addressed about the birth of any mathematical theory, and this one is no exception: the first is to account, however succinctly, for the gestation period; the second concerns the actual time and place of its birth; and the third has to do with the resonance of the event.

Firstly, the gestation period. In this case, one may want to skip the archaeological phase (1932–1947) in which the physical concepts and mathematical tools were devised that allowed the passage from a quantum probability theory for finitely many degrees of freedom, to one better adapted to the consideration of systems with infinitely many degrees of freedom. Let it suffice here to evoke the spirits of Fock, Wiener, Itô and Segal. In the most elementary sense, the question was to unify two heretofore separated domains of enquiry, the description of the *classical, stochastic* process behind the evolution equation proposed in 1908 by Langevin for the Brownian motion of a particle in a viscous stochastic medium

$$dv/dt = -\gamma\, v + \xi(t) \qquad (3.1)$$

with v the velocity, γ the coefficient of viscosity, and ξ a random function, the time average of which vanishes; and the *quantum, deterministic* evolution described by the Heisenberg equation

$$dB/dt = i\,[H, B] \qquad (3.2)$$

where B and H are self–adjoint linear operators acting on a Hilbert space, with B representing an observable, and H the Hamiltonian; equivalently, upon viewing $[H, \cdot\,]$ as a derivation, the latter may be written more intuitively in the Liouville form $dB/dt = i\,\mathcal{L}[B]$. In the modern era, the problem is brought in context by mentioning, on one side, the work of Itô, cf. Iteka and Watanabe (1981); and, on the other side, the seminal contributions of Segal (1956, 1960), Streater (1969), Araki (1970), Parthasarathy and Schmidt (1972), Hudson and Cockroft (1977), Hudson (1980), or Barnet, Streater and Wilde (1982); Accardi, Frigerio and Lewis (1982); cf. Hudson [30] who does speak with a particular authority allowing for the expression of some freely admitted personal bias; some of these may have been softened by a few additions in the above list, still incomplete, even as it is limited to the years preceding 1984 for reasons to be presently explained.

Secondly, the actual birth place of the theory of quantum stochastic processes is generally marked as the papers of Hudson and Parthasarathy [32, 33, 34]; see already [31] for an early appearance of the quantum stochastic differential equation which they will use to describe quantum diffusion.

Thirdly, the resonance. While the above line travels mainly through territories where the language of quantum physics is spoken, the waves reached quite rapidly the French coast where classical probabilists viewed the event as a mathematically genuine generalization of *their* language, requiring new techniques; one of the most influential advocates of the required reformulations is Paul-André Meyer in whose work [39] the change in perspective is explicitly implemented. Parthasarathy also found it instructive to sketch an outline of quantum stochastic calculus that nevertheless allows him to demonstrate how the familiar classical Markov processes can be realized in the quantum framework [46]. Yet, as echoes were also heard from further intellectual shores: "Quantum probability has been parodied, and indeed bitterly criticized, as an industry consisting in proving non–commutative analogues of well–known commutative theorems. In my opinion it is, or should be, nothing of the kind ..." [29]. Indeed, no one here would represent differential geometry as just generalizing Euclidean geometry to curved manifolds. Hudson continues, offering his own view of quantum stochastic theory: "It inhabits a non–commutative, quantum world which is fundamentally different and new, and which completely transcends conventional mathematical experience." One of the talents of Parthasarathy is to combine unflappable firmness, steady diligence, and a necessary dose of that Indian ingredient, patience. As already mentioned, these were recognized by the request that he present [45]. Another tribute to these talents is the creation of the flourishing school of quantum stochastic theory in India, going hand–in–hand with its perhaps more outspoken counterpart in England. Indeed, this development is in large part a deliberate consequence of Parthasarathy's dedication which is evidenced – perhaps unwittingly – by his message to Sridharan, one of the organizers of "a useful symposium which could inspire the young colleagues of Seshadri to stay in India and bring up a new generation of mathematicians" [37, p.xxv].

3.3 Kalyan Bidhan Sinha

Sinha joined Parthasarathy at ISI–Delhi in 1978. He had taken his Ph.D. in Physics (1969) from the University of Rochester, where he had joined us in 1965, fresh from a M.Sc. from the University of Delhi; his B.Sc was from Presidency College, Calcutta, 1963. I am pleased to say here again that the young faculty members of the physics department at Rochester – which I joined in 1966 – had their pick from a long parade of brilliant Indian Ph.D. students attracted there by the reputation of our eminent senior colleagues, Robert Marshak, Emil Wolf, Leonard Mandel, Bruce French, and Susumu Okubo. Some students, like Sudarshan who had worked with Marshak, helped in no small way in establishing our appeal; others, more adventurous perhaps, choose recent arrivals, and I was fortunate indeed that Kalyan approached me in the very first weeks after I had arrived in Rochester. As part of his own research programme, he attended my classes and seminars on the algebraic approach to statistical mechanics and field theory, just at the time when Marshak had commissioned me to write a book on the subject [24]; the book itself appeared after Kalyan had left, but several footnotes bring explicit witness to his early mastery, part of which, I am sure he acquired from his interaction in Rochester with a then post–doctoral associate of mine, Richard Herman.

Having finished his Ph.D. with me, Kalyan joined my own former Ph.D. advisor, J.M. Jauch in Geneva, where the second phase of his training was completed. Werner Amrein and Kalyan Sinha drove to completion the book [2] on the rigorous mathematical treatment of particle scattering in quantum mechanics, an oeuvre which Jauch had left in outline at the time of his unexpected death in 1974. After the book had appeared, and was greeted with universal acclaim, Kalyan continued along this line of research, first in various positions he occupied in Europe, North America, and Brazil, and then upon his return in India, where he started training his own graduate students.

The theory of quantum stochastic processes seemed to be pretty much set on its path when Kalyan joined Parthasarathy at ISI–Delhi. Their first papers indeed confirmed its increasing scope [48]; see also [40, 1, 49]; and for reviews, see [55] and [57] respectively; these advances were gained in part by taking further advantage of techniques in functional analysis and operator algebra theory which Kalyan had acquired in his earlier and diverse activities in mathematical physics.

As these techniques were opening new vistas in modern geometry, Parthasarathy, Sinha, their collaborators and their students were ready for the extension of quantum stochastic processes to general von Neumann algebras [27, 26] and non–commutative manifolds [13, 14, 56].

At this point, the question must be asked as to whether these beautiful mathematical constructs may have as immediate an impact on our physical understanding of this world as had their classical counterparts.

From my own vantage point in non–equilibrium statistical mechanics, I am particularly impressed by the fact that a bridge is now constructed from both ends of the problem. Traditionally, we had started from the quantum mechanics of finite systems, then taking their infinite thermodynamical limit and various mixed limits – e.g. long–time/weak coupling – to focus on the appropriate regimes. Beyond this mid–point, we aimed at obtaining – and in particular cases, we did obtain – by a projection on the proper subalgebra of relevant observables, some contractive semigroups of completely positive maps, such as those encountered in the actual phenomenological world, e.g. those described by the Bloch equation in quantum spin relaxation or electric conductivity.

Thus, those of us who are involved in trying to obtain quantitative microscopic explanations of transport phenomena would be happy to get from the quantum stochastic calculus some information on the mathematical structures encountered at the mid-point of the putative bridge, as this calculus starts from the semigroups and constructs their stochastic dilations in ways that can be viewed as canonical. Enthusiasts of the new canon may even speculate that travelling the road by way of stochastic dilations is all there is: what was envisaged as the 'mid–point' *may be* what the microscopic world ultimately looks like, a brand-new challenging world indeed.

3.4 Mandyam Doddamane Srinivas

The work of Srinivas offers original insights towards an approach to quantum stochastic processes, although the name of some of the precursors – notably J.T. Lewis – appeared also in subsection 3.2 above. The motivations here are quite different, reflecting the fact that, as a physics graduate student in Rochester during the 1970s, Srinivas found himself at the confluence of two streams: quantum axiomatics and quantum optics. The first was initiated upon my own arrival in Rochester a few years before Srinivas joined us; the second was dominated on the theoretical side by an eminent pilot, Emil Wolf, and on the other side by the pioneering experiments conducted in the laboratory directed by

Leonard Mandel. Although Srinivas and I had several happy discussions together, as explicitly acknowledged in [59], Wolf – who had also directed recently the thesis of G.S. Agarwal, yet another of the brilliant Indian students in Rochester – became Srinivas' Ph.D. advisor and they wrote several papers together; cf. e.g. [63]. Srinivas completed his Ph.D. thesis in 1976.

Srinivas' own motivation to study quantum stochastic processes comes from a critique of von Neumann's description of the quantum measurement process on two accounts: (1) the traditional presentation is limited to the measurement of observables with purely discrete spectrum; and (2) no provision is made for repeated measurements, nor *a fortiori* for measurements carried on continuously in time. His critique [59] challenges the concept of so–called "experimentally verifiable propositions", and examines means to replace it by notions that reflect more closely experimental procedures such as "operations", "instruments", and "effects".

Again, as narrated in the preceding subsection, proposals such as these do not spring out of thin air; they cannot be entirely new, and they follow steps taken earlier by others. Here, I should certainly refer to the inspirations of G. Ludwig (1967), and especially to the contributions of J.T. Lewis and E.B. Davies (1969); some of the latter are systematically developed by Davies in [20], who does cite the above paper by Srinivas.

Srinivas later contributed several responses – among which [60] – to the challenges just described. For applications to the original motivation in quantum optics, cf. [62]. These, as well as other related papers by Srinivas, are collected in [61].

4 Concluding remarks

I aimed at presenting some of the circumstantial evidences for the *How and Why* of quantum statistics in India, as they do appear to an outsider; and I hope that my account can prove useful, however biased it may be by my being a mathematical physicist whose career had to unfold on foreign soils. What I see from this position is that a vital Indian school of quantum statistics has been created, enhanced by deliberate cross-fertilizations between diverse scientific disciplines. I expect that any revision of this essay in a not too distant future will be able to sight also the rising stars of quantum communication theory, and to focus more on ideas than on personalities, thus reflecting the staying power of the field rather than the genius of its early leaders.

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